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Exact Controllability and Uniform Stabilization of Kirchoff Plates with Boundary Control Only on $\Delta w|_\Sigma$ and Homogeneous Boundary Displacement

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1. INTRODUCTION, PRELIMINARIES, AND STATEMENT OF MAIN RESULTS

1.1. *Introduction and Preliminaries*

Let Ω be an open bounded domain in R^n, n typically $\geqslant 2$, with sufficiently smooth boundary Γ. In Ω we consider the following Kirchoff plate in the solution $w(t, x)$:

$$\begin{cases} w_{tt} - \gamma\, \Delta w_{tt} + \Delta^2 w = 0 & \text{in } (0, T] \times \Omega = Q; & (1.1a) \\ w(0, x) = w_0;\ w_t(0, x) = w_1 & \text{in } \Omega; & (1.1b) \\ w|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma; & (1.1c) \\ \Delta w|_{\Sigma_0} = 0 & \text{in } (0, T] \times \Gamma_0 = \Sigma_0; & (1.1d) \\ \Delta w|_{\Sigma_1} = u & \text{in } (0, T] \times \Gamma_1 = \Sigma_1, & (1.1e) \end{cases}$$

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with only one control action $u \in L_2(\Sigma_1)$ exercised in the B.C. (1.1e). This is a distinctive feature of the problem here considered over existing literature, see below. In (1.1d)–(1.1e) we have Γ_0, Γ_1 open sets of Γ; Γ_1 non-empty, $\Gamma_0 \cup \Gamma_1 = \Gamma$. Also in (1.1a), γ is a positive constant, which radically changes the dynamical behavior of the system over the case $\gamma = 0$, in that—unlike the latter—the former case $\gamma > 0$ gives rise to a hyperbolic dynamics with finite speed of propagation. In this paper we shall be concerned with the following intimately related problems for the dynamics (1.1): the issues of exact controllability and uniform stabilization in the space of optimal regularity. It will be expedient to single out the corresponding problem with homogeneous boundary data

$$\begin{cases} \phi_{tt} - \gamma\, \Delta\phi_{tt} + \Delta^2\phi = f & \text{in } Q; \quad (1.2a) \\ \phi(0, x) = \phi_0;\ \phi_t(0, x) = \phi_1 & \text{in } \Omega; \quad (1.2b) \\ \phi\,|_\Sigma \equiv 0 & \text{in } \Sigma; \quad (1.2c) \\ \Delta\phi\,|_\Sigma = 0 & \text{in } \Sigma. \quad (1.2d) \end{cases}$$

To describe our result, we shall first let A be the positive self-adjoint operator on $L_2(\Omega)$ defined by

$$\begin{aligned} A\psi &= \Delta^2\psi; & \mathscr{D}(A) &= \{\psi \in H^4(\Omega) : \psi\,|_\Gamma = \Delta\psi\,|_\Gamma = 0\}, \\ A^{1/2}\psi &= -\Delta\psi, & \mathscr{D}(A^{1/2}) &= H^2(\Omega) \cap H_0^1(\Omega). \end{aligned} \tag{1.3}$$

The following space identifications are known (with equivalent norms) [G.1]:

$$\begin{aligned} \mathscr{D}(A^\theta) &= \{\psi \in H^{4\theta}(\Omega) : \psi\,|_\Gamma = 0\}, & \tfrac{1}{8} < \theta < \tfrac{5}{8}; \\ \mathscr{D}(A^\theta) &= \{\psi \in H^{4\theta}(\Omega) : \psi\,|_\Gamma = \Delta\psi\,|_\Gamma = 0\}, & \tfrac{5}{8} < \theta \leqslant 1. \end{aligned} \tag{1.4}$$

The following specializations thereof will be needed below:

$$\theta = \frac{1}{4} : \mathscr{D}(A^{1/4}) = H_0^1(\Omega); \qquad \text{for} \quad g \in H_0^1(\Omega)$$

$$\|g\|_{\mathscr{D}(A^{1/4})} = \|A^{1/4} g\|_{L_2(\Omega)} = \left\{ \int_\Omega |\nabla g|^2\, d\Omega \right\}^{1/2}, \qquad \text{equivalent to the } \|g\|_{H_0^1(\Omega)}\text{-norm,} \tag{1.5a}$$

in turn equivalent to

$$\left\{ \int_\Omega g^2 + \gamma\, |\nabla g|^2\, d\Omega \right\}^{1/2} = \{ \|g\|^2_{L_2(\Omega)} + \gamma\, \|A^{1/2} g\|^2_{L_2(\Omega)} \}^{1/2}$$
$$= \|g\|_{\mathscr{D}(A_\gamma^{1/4})}, \tag{1.5b}$$

the latter norm being denoted by $\mathscr{D}(A_\gamma^{1/4})$-norm, or $H^1_{0,\gamma}(\Omega)$-norm;

$$\theta=\frac{1}{2}: \mathscr{D}(A^{1/2})=\{\psi\in H^2(\Omega): \psi|_\Gamma=0\}; \qquad \text{for} \quad g\in\mathscr{D}(A^{1/2})$$

$$\|g\|^2_{\mathscr{D}(A^{1/2})}\equiv\|A^{1/2}g\|^2_{L_2(\Omega)}=\int_\Omega (\Delta g)^2\,d\Omega \tag{1.6a}$$

equivalent to

$$\|A^{1/4}g\|^2_{L_2(\Omega)}+\gamma\,\|A^{1/2}g\|^2_{L_2(\Omega)}\equiv\|g\|^2_{\mathscr{D}(A_\gamma^{1/2})}, \tag{1.6b}$$

the latter norm being denoted by $\mathscr{D}(A_\gamma^{1/2})$-norm;

$$\theta=\frac{3}{4}: \mathscr{D}(A^{3/4})=\{\psi\in H^3(\Omega): \psi|_\Gamma=\Delta\psi|_\Gamma=0\}; \qquad \text{for} \quad g\in\mathscr{D}(A^{3/4})$$

$$\|g\|_{\mathscr{D}(A^{3/4})}=\|A^{3/4}g\|_{L_2(\Omega)}=\|A^{1/4}\,\Delta g\|_{L_2(\Omega)}=\left\{\int_\Omega |\nabla(\Delta g)|^2\,d\Omega\right\}^{1/2} \tag{1.7}$$

by (1.3) and (1.5a). The basic space for the solutions of problem (1.1) with $u\in L_2(\Sigma_1)$ will be the space

$$Z=\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})=[H^2(\Omega)\cap H^1_0(\Omega)]\times H^1_0(\Omega). \tag{1.8}$$

1.2. *Regularity and Exact Controllability*

We begin by stating the main (optimal) regularity results for the solutions of problems (1.1) and (1.2).

THEOREM 1.1 (Fundamental Trace Regularity for the Homogeneous Problem (1.2)). *With reference to problem* (1.2) *with* $f=0$, *we have for any* $0<T<\infty$,

$$\int_\Sigma \left[\left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2+\left(\frac{\partial\phi_t}{\partial\nu}\right)^2\right] d\Sigma\leqslant CTE_\phi(0). \tag{1.9}$$

$$E_\phi(0)\equiv\int_\Omega |\nabla(\Delta\phi_0)|^2+|\nabla\phi_1|^2+\gamma\,|\Delta\phi_1|^2\,d\Omega \tag{1.10}$$

$$=\|\phi_0\|^2_{\mathscr{D}(A^{3/4})}+\|\phi_1\|^2_{\mathscr{D}(A^{1/4})}+\gamma\,\|\phi_1\|^2_{\mathscr{D}(A^{1/2})} \tag{1.11}$$

$$=\|\phi_0\|^2_{\mathscr{D}(A^{3/4})}+\|\phi_1\|^2_{\mathscr{D}(A_\gamma^{1/2})} \qquad \textit{equivalent to } \|\{\phi_0,\phi_1\}\|^2_{\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})} \tag{1.12}$$

by (1.5), (1.6), (1.7).

By duality, or transposition, on Theorem 1.1, one obtains

THEOREM 1.2 (Interior Regularity for Problem (1.1)). *With reference to problem* (1.1), *we have that for any finite T and, say, $\Gamma_1 = \Gamma$,*

$$\{w_0, w_1, u\} \to \{w(t), w_t(t)\}: \textit{continuous } \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4}) \times L_2(0, T; L_2(\Gamma)) \\ \to C([0, T]; \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})). \tag{1.13}$$

The map in (1.13) is, in fact, surjective for T sufficiently large. Indeed, by time reversal of problem (1.1), even more is contained in the following result, which does not require geometrical conditions if $\Gamma_1 = \Gamma$.

THEOREM 1.3 (Exact Controllability on $Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})$). (a) *Let $\Gamma_1 = \Gamma$. Then there exists a time $T_0 > 0$ (which can be explicitly estimated in the proof, see* (4.26) *below) such that if $T > T_0$, then, given any $\{w_0, w_1\} \in Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})$, there exists a suitable control $u \in L_2(0, T; L_2(\Gamma))$ such that the corresponding solution of* (1.1) *satisfies*

$$w(T, \cdot) = w_t(T, \cdot) = 0; \qquad \{w, w_t\} \in C([0, T]; \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})).$$

(b) *More generally, let $\Gamma_0 \neq \phi$. Then, the same exact controllability result as in* (a) *holds true with control $u \in L_2(0, T; L_2(\Gamma_1))$, provided that:*

(b$_1$) *there exists a vector field $h(x) = [h_1(x), ..., h_n(x)] \in [C^2(\bar{\Omega})]^n$ such that*

$$\text{(i)} \quad h(x) \cdot \nu(x) \leqslant 0 \textit{ on } \Gamma_0; \ \nu = \textit{outward unit normal}; \tag{1.14}$$

$$\text{(ii)} \quad \int_\Omega H(x)\, v(x) \cdot v(x)\, d\Omega \geqslant \rho \int_\Omega |v(x)|^2_{R^n}\, d\Omega \\ \textit{for some constant } \rho > 0, \ \forall v \in [L_2(\Omega)]^n, \tag{1.15}$$

guaranteed by $H(x) + H^*(x)$ strictly positive definite on $\bar{\Omega}$, where

$$H(x) = \begin{vmatrix} \dfrac{\partial h_1}{\partial x_1} \cdots \dfrac{\partial h_1}{\partial x_n} \\ \cdot \qquad \cdot \\ \dfrac{\partial h_n}{\partial x_1} \cdots \dfrac{\partial h_n}{\partial x_n} \end{vmatrix}. \tag{1.16}$$

(b$_2$) *The corresponding homogeneous problem* (1.2a)–(1.2d), *along with the additional third B.C. $(\partial(\Delta\phi)/\partial\nu)|_{\Sigma_1} = 0$ on $\Sigma_1 = (0, T] \times \Gamma_1$ implies $\phi \equiv 0$.*

Remark 1.1. Of course, if Γ_0 is empty, then we may take $h(x) = x - x_0$, $x_0 \in R^n$, to satisfy (1.15) with $H(x) \equiv$ identity, $\rho = 1$. Moreover, in this case, the uniqueness property described in (b_2) holds true as well: see the proof of the subsequent Lemma 4.2, where we shall ultimately fall, after a change of variable, into a recent uniqueness result in [L-L.1, p. 127], whose application, however, will require in our case $\Gamma_1 = \Gamma$. We have not investigated directly whether the uniqueness property in (b_2) holds true also for $\Gamma_1 \subsetneq \Gamma$. At any rate, case (a) is contained in case (b). We shall accordingly work with a general vector field as in (i)–(ii) above.

1.3. *The Feedback System and Uniform Stabilization*

Uniform Stabilization of Problem (1.1) *by Means of an Algebraic Riccati Operator.* As a consequence of the regularity Theorem 1.2 and of the exact controllability Theorem 1.3, problem (1.1) satisfies the abstract framework of the linear quadratic regulator problem over an infinite horizon and corresponding algebraic Riccati equation, as presented in [F-L-T.1] (which extends to abstract spaces the treatment of [L-T.8] for wave equation with Dirichlet control). In particular, exact controllability on Z guarantees that the corresponding Finite Cost Condition is satisfied on Z. As a consequence, [F-L-T.1] provides a feedback operator $-B^*P[w, w_t]$ based on the Riccati operator P acting on the full pair $\{w, w_t\}$ which produces uniform exponential decay (uniform stabilization) of the Kirchoff plate (1.1) in the norm $\mathscr{L}(Z)$ of Z. See [L-T.10] for more details.

Uniform Stabilization by an Explicit, Dissipative Feedback Operator. This issue is, in fact, the main goal of the present paper. The regularity and exact controllability results in Theorems 1.1 and 1.3 indicate that the appropriate state space for stabilization is the space $\mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})$, suitably topologized. To see how it should be topologized, we first let $u \equiv 0$ in (1.1e). It then follows, by multiplying Eq. (1.1a) by w_t and integrating by parts, that the following "energy" of the *free* system is constant:

$$E_w(t) \equiv \int_\Omega (\Delta w(t))^2 + w_t^2(t) + \gamma\, |\nabla w_t(t)|^2 \, d\Omega \tag{1.17}$$

$$= \|A^{1/2} w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \gamma\, \|A^{1/4} w_t(t)\|^2_{L_2(\Omega)} \tag{1.18}$$

$$\equiv \|w(t)\|^2_{\mathscr{D}(A^{1/2})} + \|w_t(t)\|^2_{\mathscr{D}(A_\gamma^{1/4})} \equiv \text{const} = E_w(0), \tag{1.19}$$

where, as noted before, $\mathscr{D}(A_\gamma^{1/4})$, or likewise $H^1_{0,\gamma}(\Omega)$, will be our notation for the Hilbert space $\mathscr{D}(A^{1/4})$, when endowed with norm as in (1.15b)—equivalent to (1.5a)—and corresponding inner product,

$$(x, y)_{\mathscr{D}(A_\gamma^{1/4})} \equiv ((I + \gamma A^{1/2})x, y)_{L_2(\Omega)}, \qquad x, y \in \mathscr{D}(A^{1/4}) = H^1_0(\Omega). \tag{1.20}$$

Thus, the corresponding dynamic operator ($\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix}$ in the notation below) of the *free* system is skew-adjoint on the space

$$Z_\gamma \equiv \mathscr{D}(A^{1/2}) \times \mathscr{D}(A_\gamma^{1/4}), \tag{1.21}$$

or, equivalently, the s.c. group $\exp(\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix} t)$ generated by it is unitary on such space. Then, consistently with the function spaces of the exact controllability Theorem 1.3, we shall study the question of existence and construction of an explicit boundary operator $\mathscr{F}$ based on the "velocity" w_t,

$$w_t \in \mathscr{D}(A_\gamma^{1/4}) \to \mathscr{F}(w_t) \in L_2(0, \infty; L_2(\Gamma_1)), \tag{1.22}$$

such that the boundary feedback function $u = \mathscr{F}(w_t)$ once inserted in (1.1e) produces an s.c. (feedback) semigroup on Z_γ in (1.21) *which decays in the strong, respectively, uniform norm* $\mathscr{L}(Z_\gamma)$ of Z_γ, in the latter case by necessity in an exponential way. (The feedback $-B^*P[w, w_t]$ based on the Riccati operator P, referred to before, acts instead on the full pair $\{w, w_t\}$.)

Choice of the Operator $\mathscr{F}$. It is justified in Section 5 that the following choice of $\mathscr{F}$,

$$\Delta w|_{\Sigma_1} = u = \mathscr{F}(w_t) = -\left.\frac{\partial w_t}{\partial \nu}\right|_{\Sigma_1} = -\tilde{G}_2^* A w_t = \tilde{D}^* A^{1/2} w_t, \tag{1.23}$$

where the operators $\tilde{G}_2$ and $\tilde{D}$ are defined in Section 2 below, Eqs. (2.7)–(2.11), provides a reasonable candidate for the stabilization problem. Thus, the resulting candidate feedback system, whose stability properties we shall investigate, is, by (1.23),

$$\begin{cases} w_{tt} - \gamma\, \Delta w_{tt} + \Delta^2 w = 0 & \text{on} \quad (0, T] \times \Omega = Q; & (1.24\text{a}) \\ w(0, x) = w_0;\ w_t(0, x) = w_1 & \text{on } \Omega; & (1.24\text{b}) \\ w|_\Sigma \equiv 0 & \text{on} \quad (0, T] \times \Gamma \equiv \Sigma; & (1.24\text{c}) \\ \Delta w|_{\Sigma_0} = 0 & \text{on} \quad (0, T] \times \Gamma_0 \equiv \Sigma_0; & (1.24\text{d}) \\ \Delta w|_{\Sigma_1} = -\dfrac{\partial w_t}{\partial \nu} & \text{on} \quad (0, T] \times \Gamma_1 \equiv \Sigma_1. & (1.24\text{e}) \end{cases}$$

By use of the techniques explained in Section 2, see (2.7), and in (5.1), (5.2), problem (1.24) can be rewritten more conveniently in abstract form as (the sub-index F stands for "feedback")

$$w_{tt} = \mathscr{A} w + \mathscr{A} \tilde{G}_2 \tilde{G}_2^* A w_t; \qquad \frac{d}{dt} \begin{vmatrix} w \\ w_t \end{vmatrix} = \mathscr{A}_F \begin{vmatrix} w \\ w_t \end{vmatrix} \text{ on } Z_\gamma = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A_\gamma^{1/4}); \tag{1.25}$$

$$\mathscr{A}_F = \begin{vmatrix} 0 & I \\ \mathscr{A} & \mathscr{A} \tilde{G}_2 \tilde{G}_2^* A \end{vmatrix}; \qquad \mathscr{D}(\mathscr{A}_F) = \{z \in Z_\gamma : \mathscr{A}_F z \in Z_\gamma\}, \tag{1.26}$$

where $\mathscr{A} = -(I+\gamma A^{1/2})^{-1} A$ is, by (1.20), a negative self-adjoint operator on $\mathscr{D}(A_\gamma^{1/4})$.

THEOREM 1.4. (i) *(Well-posedness on Z_γ). The operator $\mathscr{A}_F$ in* (1.26) *is dissipative on $Z_\gamma = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A_\gamma^{1/4})$ (see* (1.20)) *and satisfies here range $(\lambda I - \mathscr{A}_F) = Z_\gamma$ for $\lambda > 0$. Thus, by the Lumer–Phillips theorem, $\mathscr{A}_F$ generates a strongly continuous contraction semigroup $e^{\mathscr{A}_F t}$ on Z_γ. The resolvent operator $R(\lambda, \mathscr{A}_F)$ is given by*

$$R(\lambda, \mathscr{A}_F) = \begin{vmatrix} \dfrac{I - V^{-1}(\lambda)}{\lambda} & -V^{-1}(\lambda)\mathscr{A}^{-1} \\ -V^{-1}(\lambda) & -\lambda V^{-1}(\lambda)\mathscr{A}^{-1} \end{vmatrix}; \tag{1.27}$$

$$V(\lambda) = I + \tilde{G}_2 \tilde{G}_2^* A - \lambda^2 \mathscr{A}^{-1}; \qquad \mathscr{A}^{-1} = -A^{-1}(I+\gamma A^{1/2}), \tag{1.28}$$

and is compact on Z_γ for $\operatorname{Re}\lambda > 0$. *Moreover, $0 \in \rho(\mathscr{A}_F)$, the resolvent set of $\mathscr{A}_F$.*

(ii) *(L_2-boundedness of feedback operator). For $\{w_0, w_1\} \in Z_\gamma$, we have for problem* (1.24), *with $E_w(t)$ defined by* (1.17)–(1.19),

$$\begin{aligned} \frac{dE_w(t)}{dt} = \frac{d}{dt} \left\| e^{\mathscr{A}_F t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_{Z_\gamma}^2 &= -2\, \|D^* A^{1/2} w_t\|_{L_2(\Gamma_1)}^2 \\ &= -2 \int_{\Gamma_1} \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Gamma; \end{aligned} \tag{1.29}$$

$$\begin{aligned} E_w(t) - E_w(0) &= -2 \int_0^t \int_{\Gamma_1} \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Gamma\, dt \\ &= -2 \int_0^t \|\tilde{D}^* A^{1/2} w_t\|_{L_2(\Gamma_1)}^2\, dt; \end{aligned} \tag{1.30}$$

$$\int_0^\infty \int_{\Gamma_1} \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Gamma\, dt = \int_0^\infty \|\tilde{D}^* A^{1/2} w_t\|_{L_2(\Gamma_1)}^2\, dt \leqslant E_w(0). \tag{1.31}$$

For the uniform stabilization problem, we shall take $\Gamma_0 = \emptyset$, and so $\Gamma_1 = \Gamma$ throughout.

THEOREM 1.5 (Uniform Stabilization on Z_γ). *Let $\Gamma_0 = \emptyset$ in* (1.24d). *Let Ω satisfy the following condition: There exists a vector field $h(x) \in [C^2(\bar{\Omega})]^n$ such that:*

(i) *$h(x)$ is parallel to the (unit) normal vector $\nu(x)$ on Γ;* (1.32)

(ii) *h satisfies condition* (1.15). (1.33)

Then, there exist constants $\delta > 0$ *and* $M = M_\delta \geqslant 1$ *such that the solutions of the feedback problem* (1.24) *satisfy*

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_{Z_\gamma} = \left\| e^{\mathscr{A}_F t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_{Z_\gamma} \leqslant M e^{-\delta t} \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_{Z_\gamma}, \qquad t \geqslant 0, \tag{1.34}$$

where Z_γ *is defined in* (1.21).

Remark 1.2. The class of domains Ω to which Theorem 1.5 is applicable, i.e., satisfying conditions (i) = (1.32) and (ii) = (1.33) includes the class of strictly convex domains, or set theoretic differences thereof [L-T.7]. For information on estimates of δ, we refer to [B-T.1, Remark 1.1, p. 51] or [O-T.1, Remark 1.4, p. 282].

1.4. *Literature*

References [L-L.1, Lag.1] consider Kirchoff plates. More precisely, [L-L.1, Chap. v] gives exact controllability results either under different boundary conditions, or else with the same boundary conditions $w|_\Sigma = v_0$, $\Delta w|_\Sigma = v_1$, but with the use of both controls v_0, v_1, which moreover are taken in different spaces. Reference [Lag.1, Sect. 4.4] gives a uniform stabilization result, again using different (higher) boundary conditions than the ones used here, and moreover, with use of two feedback controls. It is well known that the problems of exact controllability/uniform stabilization are much dependent on the type of B.C. and that, moreover, the presence of only one control action of the two potentially available introduces additional difficulties. It is interesting to compare the results of the present paper with those in [L.T-3] for the corresponding Euler–Bernoulli problem (which is not hyperbolic) obtained by simply setting $\gamma = 0$ in (1.1a). In [L-T.3] exact controllability is obtained in $\mathscr{D}(A^{1/2}) \times L_2(\Omega)$ using controls of class $L_2(0, T; H^{1/2}(\Gamma))$ only in $\Delta w|_\Sigma$, and with no geometrical conditions on Ω. Compare with our present Theorem 1.3. On the other hand, uniform stabilization is obtained for the Euler–Bernoulli problem under geometrical conditions far more severe than the ones given here in our Theorem 1.5 for the corresponding result. Moreover, the two problems entail different technical tools, e.g., different multipliers.

2. Operator Models for Problems (1.1) and (1.2)

Boundary Homogeneous Problem (1.2). The abstract equation which describes the homogeneous problem (1.2) is readily seen to be (Appendix C)

$$\phi_{tt} = \mathscr{A}\phi + (I + \gamma A^{1/2})^{-1} f, \qquad \phi(0) = \phi_0,\ \phi_t(0) = \phi_1; \tag{2.1}$$

$$\mathscr{A} = -(I + \gamma A^{1/2})^{-1} A; \qquad \mathscr{D}(\mathscr{A}) = \mathscr{D}(A). \tag{2.2}$$

The operator $\mathscr{A}$, which is readily seen to be a bounded perturbation of the negative operator $-A^{1/2}/\gamma$ (Appendix C), is the generator of a cosine operator $\mathscr{C}(t)$ with $\mathscr{S}(t)=\int_0^t \mathscr{C}(\tau)\,d\tau$. Thus, the solution of problem (1.2), equivalently of (2.1), is given explicitly by

$$\phi(t)=\mathscr{C}(t)\phi_0+\mathscr{S}(t)\phi_1+(Kf)(t); \tag{2.3}$$

$$\phi_t(t)=\mathscr{C}(t)\phi_1+\mathscr{A}\mathscr{S}(t)\phi_0+\frac{d(Kf)}{dt}(t); \tag{2.4}$$

$$(Kf)(t)=\int_0^t \mathscr{S}(t-\tau)(I+\gamma A^{1/2})^{-1} f(\tau)\,d\tau:$$

$$\text{continuous } L_1(0, T; [\mathscr{D}(A^{1/2})]') \to C([0, T]; \mathscr{D}(A^{1/2})); \tag{2.5}$$

$$\left(\frac{dKf}{dt}\right)(t)=\int_0^t \mathscr{C}(t-\tau)(I+\gamma A^{1/2})^{-1} f(\tau)\,d\tau:$$

$$\text{continuous } L_1(0, T; [\mathscr{D}(A^{1/2})]') \to C([0, T]; L_2(\Omega)). \tag{2.6}$$

Other continuity results for K, dK/dt can be readily given; e.g.,

$$\left\{K, \frac{dK}{dt}\right\}: L_1(0, T; L_2(\Omega)) \to C([0, T]; \mathscr{D}(A)\times\mathscr{D}(A^{1/2})).$$

Non-homogeneous Problem (1.1). By proceeding as in [L-T.1, L-T.5, T.1], we have that the abstract differential equation in factor, respectively, additive form which models problem (1.1), is given by

$$w_{tt}=\mathscr{A}(w-\tilde{G}_2 u) \qquad \text{on } L_2(\Omega);$$

or (2.7)

$$w_{tt}=\mathscr{A}w-\mathscr{A}\tilde{G}_2 u \qquad \text{on, say, } [\mathscr{D}(A)]'.$$

In (2.7), $\tilde{G}_2$ is the Green operator defined by

$$y=\tilde{G}_2 g_2 \Leftrightarrow \{\Delta^2 y=0 \text{ in } \Omega;\ y=0 \text{ on } \Gamma;\ \Delta y=0 \text{ on } \Gamma_0;\ \Delta y=g_2 \text{ on } \Gamma_1\}. \tag{2.8}$$

We readily have for future use [L-T.1, Remark 3.2; L-T.2],

$$\begin{aligned}&\tilde{G}_2=-A^{-1/2}\tilde{D};\\ &y=\tilde{D}v \Leftrightarrow \{\Delta y=0 \text{ in } \Omega;\ y=0 \text{ on } \Gamma_0;\ y=v \text{ on } \Gamma_1\},\end{aligned} \tag{2.9}$$

and by (2.2) and (2.9) and [L-T.1, Lemma 3.1; L-T.2],

$$\tilde{G}_2^* A\psi = -\tilde{D}^* A^{1/2}\psi = \begin{cases} 0 & \text{on } \Gamma_0; \\ \dfrac{\partial\psi}{\partial\nu} & \text{on } \Gamma_1. \end{cases} \tag{2.10}$$

When we take $\Gamma_0 = \varnothing$ as in the uniform stabilization problem, we shall use the notation

$$\text{for } \Gamma_0 = \varnothing\text{: } G_2 = \tilde{G}_2; \qquad D = \tilde{D}. \tag{2.11}$$

In (2.10) we have used the adjoint operators,

$$\begin{aligned} (\tilde{D}v, y)_{L_2(\Omega)} &= (v, \tilde{D}^* y)_{L_2(\Gamma)}; \\ (\tilde{G}_2 v, y)_{L_2(\Omega)} &= (v, \tilde{G}_2^* y)_{L_2(\Gamma)}. \end{aligned} \tag{2.12}$$

The following elliptic regularity properties [L-M.1] will be needed below:

$$D\text{: continuous } H^s(\Gamma) \to H^{s+1/2}(\Omega), \qquad s \text{ real}; \tag{2.13}$$

$$D^*\text{: continuous } H^{-s}(\Omega) \to H^{-s+1/2}(\Gamma), \qquad 0 \leqslant s \leqslant \tfrac{1}{2}, \tag{2.14}$$

where (2.14) follows by duality from (2.13) used in the range $0 \leqslant s \leqslant \frac{1}{2}$. In particular, (2.14) for $s=0$ followed by (2.13) for $s=\frac{1}{2}$ yields

$$DD^*\text{: continuous } L_2(\Omega) \to H^1(\Omega). \tag{2.15}$$

In our study of regularity (Section 3) and of exact controllability (Section 4), we shall need to write the solution at time $t=T$ of the non-homogeneous problem (1.1) with, say, $w_0 = w_1 = 0$, which is given explicitly by [L-T.1–L-T.9, T.1],

$$\left| \begin{matrix} w(T; t=0; w_0 = w_1 = 0) \\ w_t(T; t=0; w_0 = w_1 = 0) \end{matrix} \right| = \mathscr{L}_T u = \left| \begin{matrix} -\mathscr{A} \int_0^T \mathscr{S}(T-t)\, \tilde{G}_2 u(t)\, dt \\ -\mathscr{A} \int_0^T \mathscr{C}(T-t)\, \tilde{G}_2 u(t)\, dt \end{matrix} \right|, \tag{2.16}$$

and characterize the (Hilbert space) adjoint $\mathscr{L}_T^*$ of the above operator $\mathscr{L}_T$.

$$\left(\mathscr{L}_T u, \left| \begin{matrix} z_1 \\ z_2 \end{matrix} \right| \right)_Z = \left(u, \mathscr{L}_T^* \left| \begin{matrix} z_1 \\ z_2 \end{matrix} \right| \right)_{L_2(\Sigma)}, \qquad Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4}), \tag{2.17}$$

in terms of the corresponding homogeneous partial differential equation.

LEMMA 2.1. *For* $\{z_1, z_2\} \in Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})$, *we have*

$$\left(\mathscr{L}_T^* \left| \begin{matrix} z_1 \\ z_2 \end{matrix} \right| \right)(t) = \begin{cases} \dfrac{\partial}{\partial\nu} (A\phi(t))|_{\Sigma_1} & \textit{on } \Sigma_1; \\ 0 & \textit{on } \Sigma_0. \end{cases} \tag{2.18}$$

where $\phi(t) = \phi(t; \phi_0, \phi_1)$ *is the solution of the following backward problem*

$$\begin{cases} \phi_{tt} - \gamma\, \Delta\phi_{tt} + \Delta^2\phi = 0; & (2.19a) \\ \phi|_{t=T} = \phi_0;\ \phi_t|_{t=T} = \phi_1; & (2.19b) \\ \phi|_\Sigma = \Delta\phi|_\Sigma \equiv 0. & (2.19c) \end{cases}$$

$$\begin{aligned} \phi_0 &= (I + \gamma A^{1/2})^{-1} z_2 \in \mathscr{D}(A^{3/4}); \\ \phi_1 &= -(I + \gamma A^{1/2})^{-1} A^{1/2} z_1 \in \mathscr{D}(A^{1/2}), \end{aligned} \tag{2.20}$$

whose solution is explicitly

$$\phi(t) = \mathscr{C}(t - T)\phi_0 + \mathscr{S}(t - T)\phi_1. \tag{2.21}$$

Proof. From (2.16) and (1.8), we compute as usual [L-T.2, L-T.4, L-T.9], etc.,

$$\begin{aligned} &\left(\mathscr{L}_T u, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}\right)_Z \\ &\quad = -\left(\mathscr{A} \int_0^T \mathscr{S}(T - t)\, \tilde{G}_2 u(t)\, dt, Az_1\right)_{L_2(\Omega)} \\ &\qquad - \left(\mathscr{A} \int_0^T \mathscr{C}(T - t)\, \tilde{G}_2 u(t)\, dt, A^{1/2} z_2\right)_{L_2(\Omega)} \\ &\quad = -\int_0^T (u(t), \tilde{G}_2^* \mathscr{A}[\mathscr{S}(T - t) A z_1 + \mathscr{C}(T - t) A^{1/2} z_2])_{L_2(\Gamma)}; \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\left(\mathscr{L}_T^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}\right)(t) \\ &\quad = \tilde{G}_2^* \mathscr{A}[\mathscr{C}(t - T) A^{1/2} z_2 + \mathscr{S}(t - T)(-A z_1)] \\ &\quad = \begin{cases} -\dfrac{\partial}{\partial \nu} A^{1/2}[\mathscr{C}(t - T)(I + \gamma A^{1/2})^{-1} z_2 \\ \qquad + \mathscr{S}(t - T)(I + \gamma A^{1/2})^{-1}(-A^{1/2} z_1)] & \text{on } \Sigma_1 \\ 0 & \text{on } \Sigma_0 \end{cases} \end{aligned} \tag{2.23}$$

by (2.10), and (2.18) follows via (2.19)–(2.21).

The proof of Lemma 2.1 may also be given by multiplying the w-problem (1.1) by ϕ and the ϕ-problem (2.19) by w and integrating by parts. ∎

3. Regularity: Proof of (Trace) Theorem 1.1 and of Theorem 1.2

3.1. *A Fundamental Identity for Problem* (1.2)

Two of the three fundamental identities for problem (1.2) needed in this paper are used in this section. The third will be used in Section 4. We first formalize the property that problem (1.2) with $f \equiv 0$ (free system) is conservative or energy-preserving. For problem (1.2) with $f \equiv 0$, we define, in line with (1.10),

$$E_\phi(t) \equiv \int_\Omega |\nabla(\Delta\phi(t))|^2 + |\nabla\phi_t(t)|^2 + \gamma\, |\Delta\phi_t(t)|^2\, d\Omega$$

$$= \|\phi(t)\|^2_{\mathscr{D}(A^{3/4})} + \|\phi_t(t)\|^2_{\mathscr{D}(A_\gamma^{1/2})}. \tag{3.1}$$

PROPOSITION 3.1. *For problem* (1.2) *with* $f \equiv 0$, *we have*

$$E_\phi(t) \equiv \text{const} \equiv E_\phi(0), \qquad \forall t \in R. \tag{3.2}$$

Proof. One multiplies Eq. (1.2a) by $\Delta\phi_t$, integrates by parts, and uses the B.C. (1.2c)–(1.2d). ∎

PROPOSITION 3.2. *With reference to problem* (1.2), *let* $h(x)$ *be a smooth vector field on* $\bar{\Omega}$ *such that* $h = \nu$ (*outward unit normal*) *on* Γ. *Then, the following identity holds true with* H *defined in* (1.16):

$$\frac{1}{2}\int_\Sigma \left[\left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 + \left(\frac{\partial\phi_t}{\partial\nu}\right)^2\right] d\Sigma$$

$$= \int_Q H\,\nabla(\Delta\phi)\cdot\nabla(\Delta\phi)\, dQ + \int_Q H\,\nabla\phi_t\cdot\nabla\phi_t\, dQ$$

$$+ \frac{1}{2}\int_Q \{|\nabla\phi_t|^2 + \gamma(\Delta\phi_t)^2 - |\nabla(\Delta\phi)|^2\}\, \text{div}\, h\, dQ$$

$$+ \int_Q \phi_t \nabla(\text{div}\, h)\cdot\nabla\phi_t\, dQ$$

$$- [(\phi_t, h\cdot\nabla(\Delta\phi))_\Omega + \gamma(\Delta\phi_t, h\cdot\nabla(\Delta\phi))_\Omega]_0^T. \tag{3.3}$$

Proof. A sketch of the proof based on the multiplier $h\cdot\nabla(\Delta\phi)$ is provided in Appendix A: Eq. (A.7) there with $h\cdot\nu = 1$ yields the left hand side of (3.3), while the right hand side of (A.5) is precisely the right hand side of (3.3). ∎

3.2. *Completion of the Proof of Theorem* 1.1

From (3.1), the right hand side (R.H.S.) of identity (3.3) can be written, by Poincaré inequality on ϕ_t, as

$$\text{R.H.S. of (3.3)} = \mathcal{O}\left(\int_0^T E_\phi(t)\,dt + E_\phi(T) + E_\phi(0)\right) = C(T+2)\,E_\phi(0) \tag{3.4}$$

after using the energy preserving property (3.2). Thus, (1.9) follows from (3.3)–(3.4).

3.3. *Proof of Theorem* 1.2: *Interior Regularity*

We sketch a proof which adapts past reasonings to present circumstances.

Step 1. With reference to the map $\mathscr{L}_T^*$ characterized in Lemma 2.1, Eq. (2.18), Theorem 1.1 says a fortiori that

$$\mathscr{L}_T^*\text{: continuous } Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4}) \to L_2(\Sigma). \tag{3.5}$$

Then, with reference to the map $\mathscr{L}_T$ defined by (2.16), it follows from (3.5) that

$$\mathscr{L}_T\text{: continuous } L_2(\Sigma) \to Z = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4}). \tag{3.6}$$

Step 2. Since $\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix}$ is an s.c. group generator on Z, we can now apply Steps 3–4 as in the abstract proof of the Theorem in [L-T.11, p. 747] (originally given for second-order hyperbolic equations in [L-T.6] by abstract methods), where the operator $\mathscr{L}_T$ here corresponds to the operator J^* in (1.15) of [L-T.11]. We then conclude that if $w_0 = w_1 = 0$ in (1.1b), then the map

$$u \to \{w(t), w_t(t)\}\text{: continuous} \to C([0, T]; Z)$$

as desired. Finally, $[w_0, w_1] \in Z$ and $u \equiv 0$ in (1.1e) imply that the semigroup solution (or cosine/sine solution as in (2.21)) satisfies $[w(t), w_t(t)] = [\exp(\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix} t)]\ [w_0, w_1] \in C([0, T]; Z)$. Theorem 1.2 is proved.

4. Exact Controllability: Proof of Theorem 1.3

Step 1. By the regularity Theorem 1.2 the input-solution operator $\mathscr{L}_T$ defined by (2.16) is continuous $L_2(\Sigma_1) \to Z \equiv \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})$. By the time reversibility of problem (1.1), exact controllability at time T in the space $Z=\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})$ within the $L_2(\Sigma_1)$-class of controls is equivalent to surjectivity of $\mathscr{L}_T$, in turn equivalent to the property that $\mathscr{L}_T^*$ have a continuous inverse; i.e., there is $C_T>0$ such that

$$\left\| \mathscr{L}_T^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\|_{L_2(\Sigma_1)} \geqslant C_T \,\|\{z_1, z_2\}\|_Z, \tag{4.1}$$

where $\mathscr{L}_T^*$ is defined in (2.17) and characterized in Lemma 2.1, Eq. (2.18). Accordingly, an equivalent partial differential equation characterization of inequality (4.1)—and hence of exact controllability at $T<\infty$ over the space Z within the class of $L_2(\Sigma_1)$-controls u in (1.1e)—is as follows: There exists a constant $C'_T>0$ such that

$$\int_{\Sigma_1} \left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 d\Sigma \geqslant C'_T \,\|\{\phi_0, \phi_1\}\|^2_{\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})}, \tag{4.2}$$

where ϕ solves the homogeneous problem (2.19) with initial conditions as in (2.20). Note that by (2.20),

$$\|\phi_0\|_{\mathscr{D}(A^{3/4})} = \|A^{3/4}\phi_0\|_{L_2(\Omega)} = \|A^{1/2}(I+\gamma A^{1/2})^{-1} A^{1/4} z_2\|_{L_2(\Omega)}$$
$$\text{equivalent to } \|A^{1/4} z_2\|_{L_2(\Omega)} = \|z_2\|_{\mathscr{D}(A^{1/4})};$$
$$\|\phi_1\|_{\mathscr{D}(A^{1/2})} = \|A^{1/2}\phi_1\|_{L_2(\Omega)} = \|A^{1/2}(I+\gamma A^{1/2})^{-1} A^{1/2} z_1\|_{L_2(\Omega)}$$
$$\text{equivalent to } \|A^{1/2} z_1\|_{L_2(\Omega)} = \|z_1\|_{\mathscr{D}(A^{1/2})}.$$

Step 2. It remains to show if or when (4.2) holds true. This is done in the following proposition which is the key technical issue of the present exact controllability problem for (1.1).

PROPOSITION 4.1. *Under the assumptions of Theorem* 1.3, *there exists a time* $T_0>0$ (*estimated below in* (4.26)) *such that if* $T>T_0$, *then for a suitable constant* $C_T>0$, *we have*

$$\int_{\Sigma_1} \left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 d\Sigma_1 \geqslant C_T E_\phi(0);$$
$$E_\phi(0) \text{ equivalent to } \|\{\phi_0, \phi_1\}\|_{\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})}, \tag{4.3}$$

with $E_\phi(0)$ *as in* (1.10)–(1.12) *or* (3.1) *where by time reversal, we may take* ϕ *to be the solution of problem* (1.2) *with initial data* $\{\phi_0, \phi_1\}$ *as in* (2.20) *at* $t=0$.

Proof of Proposition 4.1. Step (i). Most of the proof is reported in Appendices A and B for convenience. We use Eq. (A.5), with the left hand side as in (A.7), and (B.5) inserted in the right hand side of (A.5). We obtain with H defined in (1.16),

$$\frac{1}{2}\int_\Sigma \left[\left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 + \left(\frac{\partial\phi_t}{\partial\nu}\right)^2\right] h\cdot\nu\, d\Sigma$$

$$= \int_Q H\nabla(\Delta\phi)\cdot\nabla(\Delta\phi)\, dQ + \int_Q H\nabla\phi_t\cdot\nabla\phi_t\, dQ$$

$$+\frac{1}{2}\int_Q \Delta\phi\, \nabla(\operatorname{div} h)\cdot\nabla(\Delta\phi)\, dQ$$

$$+\frac{1}{2}\int_Q \phi_t\nabla(\operatorname{div} h)\cdot\nabla\phi_t\, dQ + b_{0T} \tag{4.4}$$

$$b_{0T} = \left[\frac{1}{2}(\gamma\,\Delta\phi_t - \phi_t, \Delta\phi \operatorname{div} h)_\Omega + (\gamma\,\Delta\phi_t - \phi_t, h\cdot\nabla(\Delta\phi))_\Omega\right]_0^T. \tag{4.5}$$

Step (ii). Using assumption (1.15) on the matrix $H(x)$, we obtain for the right hand side (R.H.S.) of (4.4), $\varepsilon > 0$,

$$\text{R.H.S. of (4.4)} \geqslant (\rho-\varepsilon)\int_Q |\nabla(\Delta\phi)|^2 + |\nabla\phi_t|^2\, dQ$$

$$-\frac{G_h^2}{\varepsilon}\int_Q (\Delta\phi)^2 + \phi_t^2\, dQ + b_{0T}; \tag{4.6}$$

$$G_h = c\max_{\bar\Omega} |\nabla(\operatorname{div} h)|. \tag{4.7}$$

We also recall (B.6) (with $f \equiv 0$) in Appendix B and write

$$\int_Q |\nabla(\Delta\phi)|^2 + |\nabla\phi_t|^2\, dQ$$

$$\geqslant \frac{1}{2}\int_Q |\nabla(\Delta\phi)|^2 + |\nabla\phi_t|^2\, dQ + \frac{1}{2}\int_Q |\nabla(\Delta\phi)|^2 - |\nabla\phi_t|^2\, dQ$$

(by (B.6))

$$= \frac{1}{2}\int_Q |\nabla(\Delta\phi)|^2 + |\nabla\phi_t|^2\, dQ + \frac{1}{2}\int_Q \gamma(\Delta\phi_t)^2\, dQ$$

$$-\frac{1}{2}\left[(\gamma\,\Delta\phi_t - \phi_t, \Delta\phi)_\Omega\right]_0^T. \tag{4.8}$$

Then, using (4.8) in (4.6) and recalling $E_\phi(t)$ in (3.1) results in

$$\text{R.H.S. of } (4.4) \geqslant \frac{(\rho-\varepsilon)}{2} \int_0^T E_\phi(t)\, dt + \beta_{0T} - \frac{G_h^2}{\varepsilon} \int_Q (\Delta\phi)^2 + \phi_t^2 \, dQ \tag{4.9}$$

$$\beta_{0T} = b_{0T} - \frac{\rho-\varepsilon}{2} \left[(\gamma\, \Delta\phi_t - \phi_t, \Delta\phi)_\Omega \right]_0^T. \tag{4.10}$$

Using the conservation of $E_\phi(t)$, (3.2), we readily get

$$|\beta_{0T}| \leqslant C[E_\phi(T) + E_\phi(0)] \leqslant K_1 E_\phi(0). \tag{4.11}$$

We now use assumption (1.14) on h on the left hand side of (4.4), while on the right of (4.4) we combine (4.9) and (4.11) and use $E_\phi(t) \equiv E_\phi(0)$ to obtain

$$\begin{aligned} &\int_{\Sigma_1} \left(\frac{\partial(\Delta\phi)}{\partial\nu} \right)^2 h \cdot \nu \, d\Sigma + \int_\Sigma \left(\frac{\partial\phi_t}{\partial\nu} \right)^2 h \cdot \nu \, d\Sigma \\ &\qquad + 2\frac{G_h^2}{\varepsilon} T \{ \|\Delta\phi\|^2_{C([0,T]; L_2(\Omega))} + \|\phi_t\|^2_{C([0,T]; L_2(\Omega))} \} \\ &\qquad \geqslant [(\rho-\varepsilon)T - 2K_1]\, E_\phi(0), \end{aligned} \tag{4.12}$$

where if $h(x) = x - x_0$, then $G_h = 0$ in (4.7). In the case of a general $h(x)$, we complete the proof by absorbing all lower order terms in (4.12) by a compactness argument of the type used in other waves or plates problems [Lio.1, Lio.2, Lit.1, L-T.2–L-T.4, L-T.9] as adapted to present circumstances. In the case of a radial field where $G_h = 0$, we only need to absorb the boundary term $\partial\phi_t/\partial\nu$.

STEP (iii). LEMMA 4.2. *Under the uniqueness property* (b_2) *of Theorem* 1.3, *inequality* (4.12) *implies that there exists a positive time* $T_u = T_u(\Omega)$ *depending on* Ω (*subscript u stands for "uniqueness," see later*) *such that if* $T > T_u > 0$, *there exists a constant* C_T *such that*

$$\begin{aligned} &\int_\Sigma \left(\frac{\partial\phi_t}{\partial\nu} \right)^2 d\Sigma + \|\Delta\phi\|^2_{C([0,T]; L_2(\Omega))} + \|\phi_t\|^2_{C([0,T]; L_2(\Omega))} \\ &\qquad \leqslant C_T \int_{\Sigma_1} \left(\frac{\partial(\Delta\phi)}{\partial\nu} \right)^2 d\Sigma. \end{aligned} \tag{4.13}$$

Proof. The proof is by contradiction, as usual. Let there exist a sequence $\{\phi_n(t)\}$ of solutions to problem (2.19) with initial data $\{\phi_{n0}, \phi_{n1}\} \in \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2})$ as in (2.20) such that $(d/dt = ')$

$$\int_{\Sigma}\left(\frac{\partial\phi'_n}{\partial\nu}\right)^2 d\Sigma + \|\Delta\phi_n\|^2_{C([0,T];L_2(\Omega))} + \|\phi'_n\|^2_{C([0,T];L_2(\Omega))} \equiv 1; \tag{4.14}$$

$$\int_{\Sigma_1}\left(\frac{\partial(\Delta\phi_n)}{\partial\nu}\right)^2 d\Sigma \to 0 \quad \text{as} \quad n\to\infty. \tag{4.15}$$

Then $\{\phi_n\}$ satisfy inequality (4.12) and by (4.14), (4.15) we have that $\{\phi_{n0}, \phi_{n1}\}$ is uniformly bounded in $\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})$. Thus, for a subsequence, $\phi_{n0}\to\tilde{\phi}_0$ weakly in $\mathscr{D}(A^{3/4})$ and $\phi_{n1}\to$ some $\tilde{\phi}_1$ weakly in $\mathscr{D}(A^{1/2})$. We next consider the solution $\tilde{\phi}(t)$ of the same problem (2.19) generated by the initial data $\{\tilde{\phi}_0, \tilde{\phi}_1\}$, explicitly $\tilde{\phi}(t)=\mathscr{C}(t)\tilde{\phi}_0+\mathscr{S}(t)\tilde{\phi}_1$. Then $\{\phi_n(t), \phi'_n(t)\}\to\{\tilde{\phi}(t), \tilde{\phi}'(t)\}$ in $L^\infty(0,T;\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2}))$ weak star. Hence,

$$\{\phi_n(t), \phi'_n(t)\} \text{ uniformly bounded in } L^\infty(0,T;\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})). \tag{4.16}$$

Using the equation (2.1), $\phi''_n=(I+\gamma A^{1/2})^{-1}A\phi_n$, we obtain via (4.16),

$$\{\phi''_n(t), \phi'''_n(t)\} \text{ uniformly bounded in } L^\infty(0,T;\mathscr{D}(A^{1/4})\times L_2(\Omega)), \tag{4.17}$$

and thus a fortiori via (1.6) and (1.7),

$$\begin{aligned}&\phi_n \text{ uniformly bounded in } H^{3,3}(Q);\\ &\phi'_n \text{ uniformly bounded in } H^{2,2}(Q).\end{aligned} \tag{4.18}$$

Then, by trace theory [L-M.1, Vol. II, p. 9],

$$\frac{\partial\phi'_n}{\partial\nu} \text{ uniformly bounded in } H^{1/2,1/2}(\Sigma). \tag{4.19}$$

Then, by compactness from (4.16) [S.1] and, respectively, from (4.19), we have for a subsequence that

$$\phi_n\to\tilde{\phi} \qquad \text{strongly in } C([0,T];\mathscr{D}(A^{1/2})=H^2(\Omega)\cap H^1_0(\Omega)); \tag{4.20}$$

$$\phi'_n\to\tilde{\phi}' \qquad \text{strongly in } C([0,T];L_2(\Omega)); \tag{4.21}$$

$$\frac{\partial\phi'_n}{\partial\nu}\to\frac{\partial\tilde{\phi}'}{\partial\nu} \qquad \text{strongly in } L_2(\Sigma), \tag{4.22}$$

and by (4.20)–(4.22), as well as (4.14), we obtain

$$\int_{\Sigma}\left(\frac{\partial\tilde{\phi}'}{\partial\nu}\right)^2 d\Sigma + \|\Delta\tilde{\phi}\|^2_{C([0,T];L_2(\Omega))} + \|\tilde{\phi}'\|^2_{C([0,T];L_2(\Omega))} = 1. \tag{4.23}$$

On the other hand, $\tilde{\phi}$ given below (4.15) solves the homogeneous problem,

$$\begin{cases} \tilde{\phi}_{tt} - \gamma\, \Delta\tilde{\phi}_{tt} + \Delta^2\tilde{\phi} = 0 & \text{in } Q; & (4.24\text{a}) \\ \tilde{\phi}|_\Sigma = \Delta\tilde{\phi}|_\Sigma = 0 & \text{in } \Sigma; & (4.24\text{b}) \\ \dfrac{\partial(\Delta\tilde{\phi})}{\partial\nu} = 0 & \text{in } \Sigma_1; & (4.24\text{c}) \end{cases}$$

the latter identity (4.2c) from (4.15).

By changing variable $\psi = \Delta\tilde{\phi} = A^{1/2}\tilde{\phi}$, see (1.3), we obtain

$$\begin{cases} \psi_{tt} - \gamma\, \Delta\psi_{tt} + \Delta^2\psi = 0 & \text{in } Q; & (4.25\text{a}) \\ \psi|_\Sigma = \Delta\psi|_\Sigma = 0 & \text{in } \Sigma; & (4.25\text{b}) \\ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma_1, & (4.25\text{c}) \end{cases}$$

since by the equation $\Delta\psi = \Delta^2\tilde{\phi} = \gamma\, \Delta\tilde{\phi}_{tt} - \tilde{\phi}_{tt}$, which vanishes on Σ by (4.24b). For problem (4.25) with $\Gamma_0 = \emptyset$ and thus $\Gamma_1 = \Gamma$, the uniqueness question has a positive answer [L-L.1, p. 127]: If $T >$ some T_u, then $\psi = A^{1/2}\tilde{\phi} \equiv 0$ in Q, hence $\tilde{\phi} \equiv 0$ in Q, which provides a solution to the uniqueness question for problem (4.24). If $\Gamma_0 \neq \emptyset$, the uniqueness conclusion $\psi \equiv 0$ was assumed in (b_2), Theorem 1.3. Thus, in any case we have $\tilde{\phi} \equiv 0$ in Q. But $\tilde{\phi} \equiv 0$ in Q contradicts (4.23) and Lemma 4.2 is proved. ∎

Step (iv). Thus (4.12) and (4.13) yield (4.3) as desired for $T > T_0$ with, say,

$$T_0 = \max\left\{T_u, \frac{2K_1}{\rho}\right\}. \tag{4.26}$$

The proof of Proposition 4.1 is complete. ∎

Remark 4.1. In case of a radial field, a sharp estimate for the constant K_1 in (4.11) may be obtained by an observation as in [Lio.1–Lio.2].

Remark 4.2. We have not investigated directly if the required uniqueness property for the ϕ-problem (4.24) holds true if $\Gamma_0 \neq \emptyset$, $\Gamma_1 \subsetneq \Gamma$. Above we fall into the ψ-problem (4.25) and appeal to the uniqueness result in [L-L.1, p. 27] which requires in our case $\Gamma_1 = \Gamma$.

Remark 4.3. Inequality (4.3) with $\Gamma_1 = \Gamma$ of Proposition 4.1 implies a fortiori the following uniqueness result: If ϕ satisfies the Kirchoff equation (1.2) = (2.19) and, moreover, the three boundary conditions,

$$\phi|_\Sigma = \Delta\phi|_\Sigma = \left.\frac{\partial(\Delta\phi)}{\partial\nu}\right|_\Sigma = 0, \tag{4.27}$$

for $0 < t \leqslant T$, T sufficiently large, then in fact $\phi \equiv 0$ in Q. By contrast, a standard uniqueness result requires all four boundary conditions to be homogeneous.

Remark 4.4. When $\Gamma_1=\Gamma$, inequality (1.9) (trace regularity) and inequality (4.3) (continuous observability) imply that we can introduce a norm on the initial data of the homogeneous problem (2.19),

$$\|\{\phi_0, \phi_1\}\|_F=\left\{\int_\Sigma \left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 d\Sigma\right\}^{1/2},$$

which is equivalent to $E_\phi^{1/2}(0)$, see (1.10), which in turn is equivalent to $\|\{\phi_0, \phi_1\}\|_{\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})}$.

5. The Closed Loop System: Proof of Theorem 1.4

Step 1 (*Abstract Model for the Closed Loop System*). We follow the conceptual approach of [L-T.5–L-T.7, T.1, T.2] for wave equations and of [L-T.1–L-T.4, B-T.1] for the plate equations. We return to the (open loop) abstract model (2.7) for problem (1.1), rewritten now as a first order system

$$\frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix}=\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix}\begin{vmatrix} w \\ w_t \end{vmatrix}+\begin{vmatrix} 0 \\ -\mathscr{A}\tilde{G}_2 u \end{vmatrix}. \tag{5.1}$$

Since, as observed below (1.20), the operator $\begin{vmatrix} 0 & I \\ \mathscr{A} & 0 \end{vmatrix}$ is skew-adjoint on the space Z_γ (defined in (1.21)), Eq. (5.1) plainly suggests to take

$$\Delta w|_{\Sigma_1}=u=-\tilde{G}_2^* A w_t=\tilde{D}^* A^{1/2} w_t=-\frac{\partial w_t}{\partial\nu}\bigg|_{\Sigma_1}, \tag{5.2a}$$

so that (5.1) becomes

$$\frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix}=\mathscr{A}_F\begin{vmatrix} w \\ w_t \end{vmatrix}, \qquad \mathscr{A}_F=\begin{vmatrix} 0 & I \\ \mathscr{A} & \mathscr{A}\tilde{G}_2\tilde{G}_2^* A \end{vmatrix}, \tag{5.2b}$$

see (2.9)–(2.10) as a natural candidate for feedback stabilization as explicitly noted in (1.23), for this choice then makes the resulting feedback operator $\mathscr{A}_F$ defined in (5.2b) = (1.26) dissipative on Z_γ: recalling the inner product (1.20) on $\mathscr{D}(A_\gamma^{1/2})$ we have for $y=[y_1, y_2]\in\mathscr{D}(\mathscr{A}_F)$, and hence $u=-\tilde{G}_2^* A y_2$ by (5.2),

$$\begin{aligned}\operatorname{Re}(\mathscr{A}_F y, y)_{Z_\gamma}&=0+(\mathscr{A}\tilde{G}_2\tilde{G}_2^* A y_2, y_2)_{\mathscr{D}(A_\gamma^{1/4})}\\ &=((I+\gamma A^{1/2})\mathscr{A}\tilde{G}_2\tilde{G}_2^* A y_2, y_2)_{L_2(\Omega)} &&(5.3a)\\ &=-((I+\gamma A^{1/2})(I+\gamma A^{1/2})^{-1} A\tilde{G}_2\tilde{G}_2^* A y_2, y_2)_{L_2(\Omega)}\\ &=-\|\tilde{G}_2^* A y_2\|^2_{L_2(\Gamma)}\leqslant 0, &&(5.3b)\end{aligned}$$

where in going from (5.2) to (5.3) we have recalled the definition of $\mathscr{A}$ in (2.2). Then, with this choice for u, the resulting closed loop problem, where (1.23) = (5.2a) is inserted in (1.1e), takes the abstract form, see also (2.9),

$$w_{tt} = \mathscr{A}w + \mathscr{A}\tilde{G}_2\tilde{G}_2^* A w_t = \mathscr{A}w + \mathscr{A}A^{-1/2}\tilde{D}\,\tilde{D}^* A^{1/2} w_t \tag{5.4}$$

on say $[\mathscr{D}(A)]'$, i.e., (1.25), or the explicit partial differential equation form as in (1.24).

Step 2 (*Well Posedness*). Generation by $\mathscr{A}_F$ of an s.c. semigroup on Z_γ follows via the Lumer–Phillips theorem, since $\mathscr{A}_F$ is actually maximal dissipative and indeed direct computations as in, say, [L-T.7, L-T.3, T.3, B-T.1], which are omitted here, yield the explicit expression (1.27) for the resolvent operator, where the operator $V(\lambda)$ in (1.28) is boundedly invertible on $\mathscr{D}(A^{1/2})$ for $\lambda > 0$, since $A + \lambda A\tilde{G}_2\tilde{G}_2^* A + \lambda^2(I + \gamma A^{1/2})$ is equivalently boundedly invertible on $L_2(\Omega)$.

Step 3. The dissipativity (1.29) and the L_2-boundedness in time (1.30) follow at once from (5.3) with $y = [w(t), w_t(t)]$, since then (5.3) becomes

$$\frac{1}{2}\frac{d}{dt}\left\| e^{\mathscr{A}_F t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|^2_{Z_\gamma} = -\|\tilde{G}_2^* A w_t\|^2_{L_2(\Gamma_1)} = -\int_{\Gamma_1}\left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Gamma_1 \tag{5.5}$$

in the Z_γ-norm: integrating in t and using contraction yields (1.30), (1.31). (Indeed, once uniform stabilization is proved, the map bound of 1 in (1.31) may be replaced with the bound $\frac{1}{2}$.)

6. Uniform Stabilization: Proof of Theorem 1.5

Since now $\Gamma_0 = \varnothing$ and $\Gamma_1 = \Gamma$, we shall use throughout G_2 and D instead of $\tilde{G}_2$, $\tilde{D}$; see (2.11).

6.1. *A Change of Variable* $w \to p$

For the feedback problem (1.24), we define the "energy" $E_w(t)$ (as in (1.18)) by the squared norm of the semigroup in Theorem 1.4 on $Z_\gamma = \mathscr{D}(A^{1/2}) \times \mathscr{D}(A_\gamma^{1/4})$ (see (1.6) and (1.19), (1.20)):

$$\begin{aligned} E_w(t) &= \left\| e^{\mathscr{A}_F t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|^2_{Z_\gamma} \\ &= \left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix}\right\|^2_{Z_\gamma} = \|A^{1/2}w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{\mathscr{D}(A_\gamma^{1/4})} \\ &= \|A^{1/2}w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \gamma\,\|A^{1/4}w(t)\|^2_{L_2(\Omega)} \\ &= \int_\Omega \{(\Delta w(t))^2 + w_t^2(t) + \gamma\,|\nabla w_t(t)|^2\}\,d\Omega \leqslant E_w(0) \end{aligned} \tag{6.0}$$

by the contraction property of (1.29) in Theorem 1.4. With reference to problem (1.24), our main goal will be to show, as usual, that there exists a time $0 < T < \infty$ and a corresponding constant $c = c_T > 0$ such that

$$E_w(T) \leqslant c_T \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma, \tag{6.1}$$

for then (6.10) combined with Eq. (1.30),

$$E_w(0) = E_w(T) + 2 \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma \tag{6.2}$$

from the dissipativity relation, yields $E_w(0) \geqslant (2 + 1/c)\, E(T)$, and hence, as usual,

$$E_w(T) < E_w(0), \qquad \text{or} \qquad \|e^{\mathscr{A}_F T}\|_{\mathscr{L}(Z_\gamma)} < 1, \tag{6.3}$$

which implies the desired uniform (exponential decay (1.34)). (We refer to say [B-T.1, Remark 3.1] for a discussion of advantages and disadvantages of using criterion (6.3) over Datko's theorem. In this paper, due to the more complex dynamics, we shall use (6.3).) Adapting to present circumstances the ideas of [L-T.7, L-T.3, B-T.1] on the basis of the multipliers used in Section 3, we introduce a new variable p by setting

$$p = A^{-1/2} w_t \in C([0, T]; \mathscr{D}(A^{3/4})), \tag{6.4}$$

where the indicated regularity is a consequence of Theorem 1.4(i). Thus, by (5.4) and (6.4), we obtain since $A^{-1/2}$ and $\mathscr{A}$ in (2.2) commute

$$\begin{aligned} p_t = A^{-1/2} w_{tt} &= A^{-1/2} \mathscr{A} w + A^{-1/2} \mathscr{A} G_2 G_2^* A w_t \\ &= -(I + \gamma A^{1/2})^{-1} [A^{1/2} w + D\, D^* A^{1/2} w_t]; \end{aligned} \tag{6.5a}$$

$$p_t + \gamma A^{1/2} p_t = -[A^{1/2} w + D\, D^* A^{1/2} w_t], \tag{6.5b}$$

after using (2.2), (2.9). Thus, from (6.5),

$$p_{tt} = \mathscr{A} p - (I + \gamma A^{1/2})^{-1} A^{1/2} G_2 G_2^* A w_{tt}, \tag{6.6}$$

or

$$(I + \gamma A^{1/2})\, p_{tt} + A p = F; \tag{6.7}$$

$$\begin{aligned} F &= -A^{1/2} G_2 G_2^* A w_{tt} = -D\, D^* A^{1/2} w_{tt} = -D\, D^* A^{1/2} A^{1/2} p_t \\ &= D \frac{\partial (A^{1/2} p_t)}{\partial \nu} = -D \frac{\partial (\Delta p_t)}{\partial \nu}, \end{aligned} \tag{6.8}$$

after recalling (2.9)–(2.11), and (6.4), (1.3). In terms of the scalar function $p(t, x)$, $x \in \Omega$, corresponding to the vector-valued function $p(t) = p(t, \cdot)$, the abstract equation (6.7) can be rewritten explicitly as the following Kirchoff homogeneous problem

$$\begin{cases} p_{tt} - \gamma \Delta p_{tt} + \Delta^2 p = F & \text{in } (0, T] \times \Omega = Q; & (6.9a) \\ p(0, x) = p_0 = A^{-1/2} w_1 & \text{in } \Omega; & (6.9b) \\ p_t(0, x) = p_1 = -(I + \gamma A^{1/2})^{-1} & & \\ \qquad \times [A^{1/2} w_0 + D D^* A^{1/2} w_1] & \text{in } \Omega; & (6.9c) \\ p|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma; & (6.9d) \\ \Delta p|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma, & (6.9e) \end{cases}$$

where the homogeneous B.C. are a consequence of $p \in \mathscr{D}(A^{3/4})$, see above (1.7). As usual, it will suffice to take initial data $\{w_0, w_1\}$ in a smooth space dense in Z_γ, say in $\mathscr{D}(\mathscr{A}_F)$ with, say, $w_1 \in H_0^2(\Omega)$, which is dense in $H_0^1(\Omega)$ (the second coordinate space of the state space Z_γ in (1.21)), and obtain the inequality (6.1) with constant C_T independent of the initial data. Then, $w_1 \in H_0^2(\Omega)$ implies $D^* A^{1/2} w_1 = -\partial w_1/\partial \nu = 0$ and $p_1 = -(I + \gamma A^{1/2})^{-1} A^{1/2} w_0$ from (6.9c). In the analysis below of the p-system (6.9), we shall crucially use the following relationships between the desired norms in the original variable w and the norms in the new variable p which one obtains by use of multipliers techniques to be displayed below (same as in Section 3):

$$\|w_t\|_{\mathscr{D}(A_\gamma^{1/4})} = \{\|w_t\|^2_{L_2(\Omega)} + \gamma \|A^{1/4} w_t\|^2_{L_2(\Omega)}\}^{1/2}$$

equivalent to

$$\|A^{1/4} w_t\|_{L_2(\Omega)} = \|A^{3/4} p\|_{L_2(\Omega)} = \left\{\int_\Omega |\nabla(\Delta p)|^2 \, d\Omega\right\}^{1/2} \tag{6.10}$$

by direct use of (1.5), (6.4), and (1.7); similarly,

$$\begin{aligned} \|(I + \gamma A^{1/2}) p_t\|_{L_2(\Omega)} &= \{\|p_t\|^2_{L_2(\Omega)} + \gamma^2 \|A^{1/2} p_t\|^2_{L_2(\Omega)} + 2\gamma \|A^{1/4} p_t\|^2_{L_2(\Omega)}\}^{1/2} \\ &= \|A^{1/2} w\|_{L_2(\Omega)} + \mathcal{O}(\|D^* A^{1/2} w_t\|_{L_2(\Gamma)}) \end{aligned} \tag{6.11}$$

equivalent to

$$\begin{aligned} \|p_t\|_{\mathscr{D}(A_\gamma^{1/2})} &= \{\|A^{1/4} p_t\|^2_{L_2(\Omega)} + \gamma \|A^{1/2} p_t\|^2_{L_2(\Omega)}\} \\ &= \left\{\int_\Omega |\nabla p_t|^2 + \gamma |\Delta p_t|^2 \, d\Omega\right\}^{1/2} \end{aligned} \tag{6.12}$$

by (6.5b), (1.6), (1.5); since p satisfies the B.C. (6.9c)–(6.9d), where we recall that the feedback operator on Σ is

$$-G_2^* A w_t = D^* A^{1/2} w_t = \Delta w|_{\Sigma} = -\left.\frac{\partial w_t}{\partial \nu}\right|_{\Sigma} \tag{6.13}$$

$$= -\frac{\partial}{\partial \nu} A^{1/2} A^{-1/2} w_t = -\frac{\partial A^{1/2} p}{\partial \nu} = \frac{\partial \Delta p}{\partial \nu} \quad \text{on } \Sigma. \tag{6.14}$$

6.2. *An Identity for the p-System* (6.9)

The system (6.9) for p is of the same type as the system (2.19) for ϕ in the controllability question, with the exception of the presence of the non-homogeneous term F defined by (6.8) in the right hand side of (6.9a). It will be such a (non-smooth) term F that is the major cause of difficulties in the analysis below. We begin with an identity for p which is the counterpart of the identity (4.4) for ϕ. Henceforth, with no further mention, we take smooth initial data

$$\{w_0, w_1\} \in \mathscr{D}(\mathscr{A}_F). \tag{6.15}$$

PROPOSITION 6.1. *With reference to* (6.9), *we have the identity* (*recall* H *in* (1.16))

$$\frac{1}{2}\int_{\Sigma}\left[\left(\frac{\partial(\Delta p)}{\partial \nu}\right)^2 + \left(\frac{\partial p_t}{\partial \nu}\right)^2\right] h\cdot\nu \, d\Sigma$$

$$= \int_Q H\nabla(\Delta p)\cdot\nabla(\Delta p)\, dQ + \int_Q H\nabla p_t\cdot\nabla p_t\, dQ$$

$$+\frac{1}{2}\int_Q \Delta p\, \nabla(\operatorname{div} h)\cdot\nabla(\Delta p)\, dQ + \frac{1}{2}\int_Q p_t \nabla(\operatorname{div} h)\cdot\nabla p_t\, dQ$$

$$+\int_Q Fh\cdot\nabla(\Delta p)\, dQ + \frac{1}{2}\int_Q F\,\Delta p \operatorname{div} h\, dQ$$

$$+\left[(\gamma\,\Delta p_t - p_t, h\cdot\nabla(\Delta p))_{\Omega} + \frac{1}{2}(\gamma\,\Delta p_t - p_t, \Delta p \operatorname{div} h)_{\Omega}\right]_0^T. \tag{6.16}$$

In (6.16), the crucial term is the one involving $Fh\cdot\nabla\,\Delta p$. Accordingly, we rewrite (6.16) in a more convenient way as in the following basic identity.

PROPOSITION 6.2. *With reference to problem* (6.9), *the following identity holds true*:

$$\frac{1}{2}\int_{\Sigma}\left[\left(\frac{\partial(\Delta p)}{\partial \nu}\right)^2+\left(\frac{\partial p_t}{\partial \nu}\right)^2\right] h\cdot\nu\, d\Sigma$$

$$=\int_Q H\nabla(\Delta p)\cdot\nabla(\Delta p)\, dQ+\int_Q H\nabla p_t\cdot\nabla p_t\, dQ$$

$$+\frac{1}{2}\int_Q \Delta p\,\nabla(\operatorname{div} h)\cdot\nabla(\Delta p)\, dQ+\frac{1}{2}\int_Q p_t\nabla(\operatorname{div} h)\cdot\nabla p_t\, dQ$$

$$+\int_0^T (D\,D^*A^{1/2}w_t, h\cdot\nabla(\Delta p_t))_\Omega\, dt$$

$$+\frac{1}{2}\int_0^T (D\,D^*A^{1/2}w_t, \Delta p_t \operatorname{div} h)_\Omega\, dt$$

$$+\left[(A^{1/2}w, h\cdot\nabla(\Delta p))_\Omega+\frac{1}{2}(A^{1/2}w, \Delta p\operatorname{div} h)_\Omega\right]_0^T. \tag{6.17}$$

Proof. We recall (6.8) and integrate by parts in t,

$$\int_0^T (F, h\cdot\nabla(\Delta p))_\Omega\, dt$$

$$= -\int_0^T (D\,D^*A^{1/2}w_{tt}, h\cdot\nabla(\Delta p))_\Omega\, dt$$

$$= -\left[(D\,D^*A^{1/2}w_t, h\cdot\nabla(\Delta p))_\Omega\right]_0^T$$

$$+\int_0^T (D\,D^*A^{1/2}w_t, h\cdot\nabla(\Delta p_t))_\Omega\, dt; \tag{6.18}$$

$$\int_0^T (F, \Delta p\operatorname{div} h)_\Omega\, dt$$

$$= -\int_0^T (D\,D^*A^{1/2}w_{tt}, \Delta p\operatorname{div} h)_\Omega\, dt$$

$$= -\left[(D\,D^*A^{1/2}w_t, \Delta p\operatorname{div} h)_\Omega\right]_0^T$$

$$+\int_0^T (D\,D^*A^{1/2}w_t, \Delta p_t\operatorname{div} h)_\Omega\, dt. \tag{6.19}$$

We then insert (6.18) and (6.19) into the right hand side of (6.16) and use (6.5b), i.e., (from (1.3) since $p_t|_\Sigma=0$),

$$A^{1/2}w=\gamma\,\Delta p_t-p_t-D\,D^*A^{1/2}w_t \tag{6.20}$$

to combine the $(\ ,\ )_\Omega$-terms. This way, we obtain (6.17). ∎

6.3. *Preliminary Lower Bound Estimates for the Right Hand Side of* (6.17)

PROPOSITION 6.3. *Under the sole assumption* (1.15) *on the matrix* $H(x)$ *for the vector field* h (*and specifically with no requirement* (1.32) *that* h *be parallel to* ν *on* Γ), *we have the following inequality for the right hand side* (*R.H.S.*) *of identity* (6.17) *for any* $\varepsilon > 0$:

R.H.S. of (6.17)

$$\geqslant \frac{\rho-\varepsilon}{2} \int_Q \{|\nabla(\Delta p)|^2 + |\nabla p_t|^2 + \gamma(\Delta p_t)^2\}\, dQ$$

$$- \frac{K_{1,h}}{\varepsilon} \int_Q \{(\Delta p)^2 + p_t^2\}\, dQ - \frac{K_{2,h,\rho}}{\varepsilon\sqrt{\gamma}} \int_0^T \|D^* A^{1/2} w_t\|^2_{L_2(\Gamma)}$$

$$+ \int_0^T (D D^* A^{1/2} w_t, h \cdot \nabla(\Delta p_t))_\Omega\, dt + \beta_{0T}; \tag{6.21}$$

$$\beta_{0T} = \left[A^{1/2} w, h \cdot \nabla(\Delta p))_\Omega + \frac{1}{2} (A^{1/2} w, \Delta p \operatorname{div} h)_\Omega \right.$$

$$\left. - \frac{\rho - \varepsilon}{2} (A^{1/2} w, \Delta p)_\Omega \right]_0^T, \tag{6.22}$$

where we note explicitly that $K_{1,h} = 0$ *if* $h(x)$ *is radial* (*or linear*) ($K_{1,h}$ *is proportional to* max $|\nabla(\operatorname{div} h)|$ *over* $\bar{\Omega}$).

Proof. Step (i). We proceed as in (4.8) by use of identity (B.6) in Appendix B:

$$\int_Q |\nabla(\Delta p)|^2 + |\nabla p_t|^2$$

$$\geqslant \frac{1}{2} \int_Q |\nabla(\Delta p)|^2 + |\nabla p_t|^2\, dQ + \frac{1}{2} \int_Q |\nabla(\Delta p)|^2 - |\nabla p_t|^2\, dQ$$

(by (B.6))

$$= \frac{1}{2} \int_Q |\nabla(\Delta p)|^2 + |\nabla p_t|^2\, dQ + \frac{1}{2} \int_Q \gamma(\Delta p_t)^2\, dQ$$

$$- \frac{1}{2} \int_Q F \Delta p\, dQ - \frac{1}{2} [(\gamma \Delta p_t - p_t, \Delta p)_\Omega]_0^T$$

(by (6.19) with $\operatorname{div} h \equiv 1$)

$$= \frac{1}{2} \int_Q \{|\nabla(\Delta p)|^2 + |\nabla p_t|^2 + \gamma(\Delta p_t)^2\}\, dQ$$

$$- \frac{1}{2} \int_0^T (D D^* A^{1/2} w_t, \Delta p_t)_\Omega\, dt - \frac{1}{2} [(A^{1/2} w, \Delta p)_\Omega]_0^T, \tag{6.23}$$

after combining the (, $)_\Omega$ terms by virtue of (6.20).

Step (ii). We plainly have since D is bounded

$$\int_0^T (D\,D^*A^{1/2}w_t, \Delta p_t \operatorname{div} h)_\Omega\, dt$$

$$\geqslant -\frac{1}{\sqrt{\gamma}} \frac{\text{const}_h}{\varepsilon} \int_0^T \|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}\, dt - \varepsilon \int_0^T \gamma(\Delta p_t)^2\, dt. \quad (6.24)$$

Step (iii). We use assumption (1.15) on H for the first two terms on the right of (6.17); play $2ab < \varepsilon a^2 + (1/\varepsilon) b^2$ with the next two integral terms; use (6.23), (6.24), and a similar estimate for the last integral term in (6.23). This way we obtain (6.21). ▌

LEMMA 6.4. *With reference to β_{0T} in* (6.22), *we have for some $C > 0$ independent of T,*

$$|\beta_{0T}| \leqslant C[E_w(T) + E(0)] \leqslant C_{h\rho\varepsilon}\left[E_w(T) + \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma\right]. \quad (6.25)$$

Proof. Immediate from (6.22), (6.10), (6.0) via the Poincaré inequality (see (6.9e)) for the first inequality, and then by (6.2) for the second inequality. ▌

PROPOSITION 6.5. *Under the sole assumption* (1.15) *on the matrix $H(x)$ as in Proposition* 6.3, *we have*

R.H.S. of (6.17)

$$\geqslant \frac{\rho - \varepsilon}{2} \int_Q \{|\nabla(\Delta p)|^2 + |\nabla p_t|^2 + \gamma(\Delta p_t)^2\}\, dQ$$

$$- \frac{K_{1,h}}{\varepsilon} \int_Q \{(\Delta p)^2 + p_t^2\}\, dQ - \text{const}_{\varepsilon,\gamma,h} \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma$$

$$- C_{h,\rho,\varepsilon} E_w(T) + \int_0^T (D\,D^*A^{1/2}w_t, h \cdot \nabla(\Delta p_t))_\Omega\, dt. \quad (6.26)$$

Proof. We combine (6.25) with (6.21) with $D^*A^{1/2}w_t = \partial w_t/\partial \nu$ on Σ, see (5.2a). ▌

6.4. *Analysis of the Term in* (6.26) *Involving* $D((\Delta w)|_\Gamma) = D\,D^*A^{1/2}w_t$

Completion of the Proof of Theorem 1.5. The following estimate on the last integral on the right hand side of (6.26) is the most demanding technical issue of the present paper. It presents, with respect to the Kirchoff problem (1.1), a difficulty of the same type as the one encountered in

[L-T.7, Proposition 3.2] in the study of the uniform stabilization problem of the wave equation with feedback operator in the Dirichlet boundary conditions. It is in establishing the next result that the assumption that the vector field $h(x)$ is parallel to $\nu(x)$ on Γ is used.

THEOREM 6.6. *Let the vector field $h(x)$ be parallel to $\nu(x)$ on Γ. Then the following estimate holds true*:

$$\int_0^T (D\,D^*A^{1/2}w_t, h\cdot\nabla(\Delta p_t))_\Omega\,dt$$

$$= \mathcal{O}\left(\int_0^T \|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}\,dt + \int_0^T \|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)}\,dt\right)$$

$$+ \mathcal{O}([E_w(T)+E_w(0)]), \tag{6.27}$$

where the constants in $\mathcal{O}$ are of the form $\|D\|\,C_h/\gamma$.

Proof. The proof is given in the subsequent Section 6.5. ∎

Using Theorem 6.6 we can now complete the proof of inequality (6.1) and thus of Theorem 1.5.

PROPOSITION 6.7. *Under the assumption of Theorem 6.6, we have the estimate*

$$\frac{1}{2}\int_\Sigma \left[\left(\frac{\partial(\Delta p)}{\partial\nu}\right)^2 + \left(\frac{\partial p_t}{\partial\nu}\right)^2\right] h\cdot\nu\,d\Sigma + C_{\varepsilon\gamma h}\int_\Sigma \left(\frac{\partial(\Delta p)}{\partial\nu}\right)^2 d\Sigma$$

$$+ K_{1h\varepsilon}T\{\|\Delta p\|^2_{C([0,T];L_2(\Omega))} + \|p_t\|^2_{C([0,T];L_2(\Omega))}\}$$

$$\geqslant C_{1\rho\varepsilon\gamma h}(T - C_{2\rho\varepsilon\gamma h})\,E_w(T). \tag{6.28}$$

Proof. We insert (6.27) into the right hand side of (6.26) and recall from (6.13) that $D^*A^{1/2}w_t = -\partial w_t/\partial\nu$ on Σ and also the identity (6.11) for $A^{1/2}w$. We obtain

$$\text{R.H.S. of (6.17)} \geqslant \frac{\rho-2\varepsilon}{2}\int_Q \{|\nabla(\Delta p)|^2 + |\nabla p_t|^2 + \gamma(\Delta p_t)^2\}\,dQ$$

$$-\frac{K_{1,h}}{\varepsilon}\int_Q \{(\Delta p)^2 + p_t^2\}\,dQ - \text{const}_{\varepsilon\gamma h}\int_\Sigma \left(\frac{\partial w_t}{\partial\nu}\right)^2 d\Sigma$$

$$- C_{h\rho\varepsilon}[E_w(T)+E_w(0)]. \tag{6.29}$$

Next, to return from p to w, we recall the norm-equivalence (6.10) along with (1.5b), as well as the norm-equivalence (6.11)–(6.12). We obtain

$$\begin{aligned}
&\int_Q |\nabla(\Delta p)|^2 + |\nabla p_t|^2 + \gamma(\Delta p_t)^2 \, dQ \\
&\quad \geqslant c_\gamma \int_0^T \{\gamma \, \|A^{1/4} w_t\|^2_{L_2(\Omega)} + \|w_t\|^2_{L_2(\Omega)} + \|A^{1/2} w\|^2_{L_2(\Omega)}\} \, dt \\
&\qquad - c_\gamma \int_0^T \|D^* A^{1/2} w_t\|^2_{L_2(\Gamma)} \, dt \\
&\quad = C_\gamma \int_0^T E_w(t) \, dt - c_\gamma \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma,
\end{aligned} \tag{6.30}$$

where in the last step we have recalled (6.0) and (6.13). We now return to (6.29): we use (6.30) for the first term on the right of (6.29), and (6.2) for the last term on the right of (6.29). We obtain

$$\begin{aligned}
\text{R.H.S. of } (6.17) \geqslant (\rho - 2\varepsilon) C_\gamma \int_0^T E_w(t) \, dt - \frac{K_{1h}}{\varepsilon} \int_Q \{(\Delta p)^2 + p_t^2\} \, dQ \\
- \text{const}_{\varepsilon\gamma h} \int_\Sigma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma - C_{h\rho\varepsilon} E_w(T).
\end{aligned} \tag{6.31}$$

Finally, using the dissipativity property (from (1.29)),

$$\int_0^T E_w(t) \, dt \geqslant T E_w(T) \tag{6.32}$$

on the right of (6.31) and recalling (6.14), as well as the L.H.S. of (6.17), we obtain (6.28). ∎

Next, we "absorb" the lower-order terms in (6.28). (We are assuming $\Gamma_1 = \Gamma$; see Remark 4.2 otherwise.)

PROPOSITION 6.8. *Inequality* (6.28) *implies that for T sufficiently large, there exists $C_T > 0$ such that*

$$\begin{aligned}
&\int_\Sigma \left(\frac{\partial p_t}{\partial \nu}\right)^2 d\Sigma + \|\Delta p\|^2_{C([0, T]; L_2(\Omega))} + \|p_t\|^2_{C([0, T]; L_2(\Omega))} \\
&\qquad \leqslant C_T \int_\Sigma \left(\frac{\partial(\Delta p)}{\partial \nu}\right)^2 d\Sigma.
\end{aligned} \tag{6.33}$$

Proof. We proceed as in the proof of Lemma 4.2, with respect this time to the p-problem (6.9). We only note explicitly that when we arrive at $\partial(\Delta\tilde{p})/\partial\nu = 0$ on Σ (counterpart of (4.24b)) for the limit $\tilde{p}$, we then obtain that the right hand side of the Eq. (6.9a) for the $\tilde{p}$-problem becomes $\tilde{F} = -D(\partial\tilde{p}_t/\partial\nu) \equiv 0$ by (6.8). Thus, the $\tilde{p}$-problem is homogeneous on the right hand side, precisely like the ϕ-problem (4.24). The rest of the proof may then follow the argument of Lemma 4.2, below (4.24), and is based on the uniqueness property of the resulting $\tilde{p}$-problem (same as the $\tilde{\phi}$-problem (4.24)) to produce a contradiction. ∎

By using now (6.33) in (6.28) and recalling (6.14), we finally arrive at the sought-after inequality (6.1), that we state formally as a corollary.

COROLLARY 6.9. *Under the assumptions of Theorem* 1.5 *we obtain*

$$\int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma \geqslant C_{1\rho\varepsilon\gamma h}(T - C_{2\rho\varepsilon\gamma h})\, E_w(T), \tag{6.34}$$

and with T sufficiently large, inequality (6.1) *is proved.*

Thus, the proof of Theorem 1.5 is complete as soon as we prove Theorem 6.6.

6.5. *Proof of Theorem* 6.6: *h Parallel to* ν *on* Γ

We follow as a guideline the proof of [L-T.7, Proposition 3.2], although new technicalities are now present.

PROPOSITION 6.10. *The following estimate holds:*

$$\begin{aligned} &\int_0^T (D\,D^* A^{1/2} w_t,\, h\cdot\nabla(\Delta p_t))_\Omega\, dt \\ &\quad = -\frac{1}{\gamma}\int_0^T (A^{1/2}w,\, h\cdot\nabla(D\,D^*A^{1/2}w_t)_\Omega\, dt \\ &\qquad + \mathcal{O}\left(\int_0^T \|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}\, dt\right) \\ &\qquad + \mathcal{O}\left(\int_0^T \|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}\, \|A^{1/2}w\|_{L_2(\Omega)}\, dt\right), \end{aligned} \tag{6.35}$$

where the constants on $\mathcal{O}$ *are of the form* $\|\tilde{D}\|\, C_h/\gamma$.

Proof. Since $p_t|_\Sigma = 0$ from (6.9d), we have $\Delta p_t = -A^{1/2} p_t$, see (1.3), and from (6.5b) we obtain

$$\begin{aligned} \Delta p_t &= -A^{1/2} p_t = A^{1/2}(I+\gamma A^{1/2})^{-1} [A^{1/2} w + D\,D^* A^{1/2} w_t] \\ &= \frac{1}{\gamma} A^{1/2} w + \frac{1}{\gamma} D\,D^* A^{1/2} w_t - \frac{1}{\gamma}(I+\gamma A^{1/2})^{-1} A^{1/2} w \\ &\quad - \frac{1}{\gamma}(I+\gamma A^{1/2})^{-1} D\,D^* A^{1/2} w_t, \end{aligned} \tag{6.36}$$

since $A^{1/2}(I+\gamma A^{1/2})^{-1} = [I-(I+\gamma A^{1/2})^{-1}]/\gamma$ as in Appendix C. Using (6.36) we can write

$$(D\,D^* A^{1/2} w_t, h \cdot \nabla(\Delta p_t))_\Omega = (1)+(2)+(3)+(4). \tag{6.37}$$

$$(1) = \left(\frac{1}{\sqrt{\gamma}} D\,D^* A^{1/2} w_t, h \cdot \nabla\left(\frac{1}{\sqrt{\gamma}} \tilde{D}\,\tilde{D}^* A^{1/2} w_t\right)\right)_\Omega; \tag{6.38}$$

$$(2) = \frac{1}{\gamma}(D\,D^* A^{1/2} w_t, h \cdot \nabla(A^{1/2} w))_\Omega; \tag{6.39}$$

$$(3) = -\frac{1}{\gamma}(D\,D^* A^{1/2} w_t, h \cdot \nabla((I+\gamma A^{1/2})^{-1} A^{1/2} w))_\Omega; \tag{6.40}$$

$$(4) = -\frac{1}{\gamma}(D\,D^* A^{1/2} w_t, h \cdot \nabla((I+\gamma A^{1/2})^{-1} D\,D^* A^{1/2} w_t))_\Omega. \tag{6.41}$$

We shall use the identity (from the divergence theorem)

$$\int_\Omega \phi h \cdot \nabla\psi \, d\Omega = \int_\Gamma \phi\psi h \cdot \nu \, d\Gamma - \int_\Omega \psi h \cdot \nabla\phi \, d\Omega - \int_\Omega \phi\psi \operatorname{div} h \, d\Omega, \tag{6.42a}$$

in particular, for $\phi = \psi$,

$$\int_\Omega \psi h \cdot \nabla\psi \, d\Omega = \frac{1}{2}\int_\Gamma \psi^2 h \cdot \nu \, d\Gamma - \frac{1}{2}\int_\Omega \psi^2 \operatorname{div} h \, d\Omega. \tag{6.42b}$$

Estimate of (1). We use (6.42b) with $\psi = D\,D^* A^{1/2} w_t/\sqrt{\gamma}$. We obtain

$$\begin{aligned} (1) &= \frac{1}{2\gamma}(D\,D^* A^{1/2} w_t, (D\,D^* A^{1/2} w_t) h \cdot \nu)_\Gamma \\ &\quad - \frac{1}{2}(D\,D^* A^{1/2} w_t, D\,D^* A^{1/2} w_t \operatorname{div} h)_\Omega \\ &\leqslant \frac{\|D\|\, C_h}{\gamma} \|D^* A^{1/2} w_t\|^2_{L_2(\Gamma)}. \end{aligned} \tag{6.43}$$

Estimate of (2). We use (6.42a) with $\phi = D\,D^*A^{1/2}w_t$, hence $\phi|_\Gamma = D^*A^{1/2}w_t$; and $\psi = A^{1/2}w$, hence $\psi|_\Gamma = [A^{1/2}w\|_\Gamma] = -[\Delta w]_\Gamma = D^*A^{1/2}w_t$ by (1.3) and (1.24e), (2.10), (5.2a). We obtain

$$\begin{aligned}(2) &= \frac{1}{\gamma}(D^*A^{1/2}w_t, A^{1/2}wh\cdot\nu)_\Gamma - \frac{1}{\gamma}(A^{1/2}w, h\cdot\nabla(D\,D^*A^{1/2}w_t))_\Omega \\ &\quad - \frac{1}{\gamma}(D\,D^*A^{1/2}w_t, A^{1/2}w \operatorname{div} h)_\Omega \\ &= -\frac{1}{\gamma}(A^{1/2}w, h\cdot\nabla(D\,D^*A^{1/2}w_t))_\Omega + \mathcal{O}_1(\|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}) \\ &\quad + \mathcal{O}_2(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)}),\end{aligned} \tag{6.44}$$

with constants in $\mathcal{O}_i$ of the form C_h/γ or $\|D\|\,C_h/\gamma$, respectively.

Estimates of (3) *and* (4). These are more regular terms. The terms on the right hand side of the inner products (6.40) and (6.41) are both in $H^1(\Omega)$ (a.e. the second) [e.g., $A^{1/2}w \in L_2(\Omega)$, $(I+\gamma A^{1/2})^{-1}A^{1/2}w \in \mathcal{D}(A^{1/2}) \subset H^2(\Omega)$ by (1.6a) in the first case, while the term $D\,D^*A^{1/2}w_t$ on the left of these inner products is in $H^{1/2}(\Omega)$ a.e.]

$$(3) = \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)}); \tag{6.45}$$

$$(4) = \mathcal{O}(\|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}), \tag{6.46}$$

with constants in $\mathcal{O}$ of the same form as before in (6.44). Using the estimates (6.43)–(6.46) in (6.37) and integrating in t yields (6.35). Proposition 6.10 is proved. ∎

We finally handle the first integral term at the right side of (6.35). It is this term which presents technical difficulties similar to those encountered in the stabilization of the wave equation with Dirichlet feedback [L-T.7]. These are overcome when the vector field $h(x)$ is parallel to the normal ν on Γ.

LEMMA 6.11. *Let $\{w_0, w_1\} \in \mathcal{D}(\mathcal{A}_F)$, and let the vector field $h(x)$ be parallel to the normal unit vector ν on Γ, so that $h(\sigma) = b(\sigma)\nu$, $\sigma \in \Gamma$, for a smooth boundary function b. Then we have*

$$\begin{aligned}&(A^{1/2}w(t), h\cdot\nabla(D\,D^*A^{1/2}w_t(t)))_\Omega \\ &\quad = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu}D(b\,D^*A^{1/2}w(t)), D^*A^{1/2}w(t)\right)_\Gamma \\ &\qquad + \mathcal{O}(\|D^*A^{1/2}w_t(t)\|_{L_2(\Gamma)}\,\|A^{1/2}w(t)\|_{L_2(\Omega)}) \qquad \textit{a.e. in } t.\end{aligned} \tag{6.47}$$

Proof. *Step* 1. Recalling (1.3) and $w|_\Sigma = 0$ in (1.1c) and using Green's second theorem, we obtain (all inner products are in L_2, unless otherwise noted)

$$-(A^{1/2}w, h\cdot\nabla(D\,D^*A^{1/2}w_{tt}))_\Omega$$
$$= (\Delta w, h\cdot\nabla(D\,D^*A^{1/2}w_t))_\Omega = (1) + (2); \tag{6.48}$$

$$(1) = \left(\frac{\partial w}{\partial \nu}, h\cdot\nabla(D\,D^*A^{1/2}w_t)\right)_\Gamma$$
$$= -(D^*A^{1/2}w, h\cdot\nabla(D\,D^*A^{1/2}w_t))_\Gamma \qquad \text{(by (2.10))}; \tag{6.49}$$

$$(2) = (w, \Delta(h\cdot\nabla(D\,D^*A^{1/2}w_t)))_\Omega. \tag{6.50}$$

Step 2. We analyze (1). We claim that

$$(1) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}D(b\,D^*A^{1/2}w), D^*A^{1/2}w\right)$$
$$+ \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)} \qquad \text{a.e. in } t. \tag{6.51}$$

To prove (6.51), we use the assumption $h(\sigma) = b(\sigma)\nu$ on the vector field and rewrite (1) from (6.49) by means of Green's second theorem recalling that $Dg|_\Gamma = g$ by (2.9):

$$(1) = -\left(D^*A^{1/2}w, b\frac{\partial}{\partial \nu}(D\,D^*A^{1/2}w_t)\right)_\Gamma$$
$$= -\left(b\,D^*A^{1/2}w, \frac{\partial}{\partial \nu}(D\,D^*A^{1/2}w_t)\right)_\Gamma \tag{6.52}$$
$$= -\left(\frac{\partial}{\partial \nu}D(b\,D^*A^{1/2}w), D^*A^{1/2}w_t\right)_\Gamma, \tag{6.53}$$

where cancellations occur because of the definition of D in (2.9). Next, we compute by (6.52), (6.53),

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}D(b\,D^*A^{1/2}w), D^*A^{1/2}w\right)_\Gamma$$
$$= \left(\frac{\partial}{\partial \nu}D(b\,D^*A^{1/2}w_t), D^*A^{1/2}w\right)_\Gamma$$
$$+ \left(D^*A^{1/2}w, b\frac{\partial}{\partial \nu}(D\,D^*A^{1/2}w_t)\right)_\Gamma. \tag{6.54}$$

Using $\Delta(\beta\alpha) = \beta\,\Delta\alpha + \alpha\,\Delta\beta + 2\nabla\beta \cdot \nabla\alpha = 2\nabla(Db) \cdot \nabla(Dg)$, if $\beta = Db$, and $\alpha = Dg$ for some vector $g \in L_2(\Gamma)$, we can readily verify that $D(bg) = (Db)(Dg) - \chi$, hence

$$\text{on } \Gamma\text{: } \frac{\partial D(bg)}{\partial \nu} = b\,\frac{\partial (Dg)}{\partial \nu} + g\,\frac{\partial (Db)}{\partial \nu} - \frac{\partial \chi}{\partial \nu}, \tag{6.55}$$

where χ satisfies

$$\Delta\chi = 2\nabla(Db) \cdot \nabla(Dg) \text{ in } \Omega; \qquad \chi = 0 \text{ on } \Gamma;$$

or

$$\chi = 2A^{-1}[\nabla(Db) \cdot \nabla(Dg)]. \tag{6.56}$$

We now specialize (6.55) to the case of our interest where $g = D^*A^{1/2}w_t \in L_2(\Gamma)$ a.e. in t by (1.31). Thus, the right hand side (R.H.S.) of (6.54) becomes by (6.55),

$$\begin{aligned}
\text{R.H.S. of (6.54)} &= \left(b\,\frac{\partial}{\partial \nu}\,(D\,D^*A^{1/2}w_t),\, D^*A^{1/2}w \right)_\Gamma \\
&\quad + \left(D^*A^{1/2}w,\, b\,\frac{\partial}{\partial \nu}\,(D\,D^*A^{1/2}w_t) \right)_\Gamma \\
&\quad + \left(D^*A^{1/2}w_t\,\frac{\partial (Db)}{\partial \nu},\, D^*A^{1/2}w \right)_\Gamma \\
&\quad - \left(\frac{\partial \chi}{\partial \nu},\, D^*A^{1/2}w \right)_\Gamma \qquad\qquad (6.57) \\
&= 2\left(b\,\frac{\partial}{\partial \nu}\,(D\,D^*A^{1/2}w_t),\, D^*A^{1/2}w \right)_\Gamma \\
&\quad + \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)}), \qquad (6.58)
\end{aligned}$$

since with $g = D^*A^{1/2}w_t \in L_2(\Gamma)$ a.e., $\nabla(Dg) \in H^{-1/2-\varepsilon}(\Omega)$ [L-M.1, p. 85], we obtain $\chi \in H^{3/2-\varepsilon}(\Omega)$ by (6.56), hence $\partial\chi/\partial\nu \in H^{-\varepsilon}(\Gamma)$ [K.1, Theorem 3.8.1]; on the other hand, $D^*A^{1/2}w \in H^{1/2}(\Gamma)$ by (2.9) with $s = 0$, and thus

$$\left(\frac{\partial \chi}{\partial \nu},\, D^*A^{1/2}w \right)_\Gamma = \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\,\|A^{1/2}w\|_{L_2(\Omega)}) \qquad \text{a.e. in } t, \tag{6.59}$$

which completes the proof of the step from (6.57) to (6.58). (The validity of (6.59) can be proved also by the use of Green's second theorem followed by identity (6.42a).) Then (6.54) and (6.58), along with (6.52), yield (6.51) as desired.

Step 3. We analyze (2). We claim that

$$\begin{aligned}(2) &= (w, \Delta(h \cdot (D\, D^* A^{1/2} w_t)))_\Omega \\ &= \mathcal{O}(\|D^* A^{1/2} w_t\|_{L_2(\Gamma)} \|A^{1/2} w\|_{L_2(\Omega)}) \qquad \text{a.e. in } t. \end{aligned} \tag{6.60}$$

This follows by writing

$$(2) = (A^{-1/2} A^{1/2} w, \Delta(h \cdot \nabla(D\, D^* A^{1/2} w_t)))_\Omega, \tag{6.61}$$

with $A^{1/2} w \in L_2(\Omega)$, $D^* A^{1/2} w_t \in L_2(\Gamma)$ a.e. in t and proceeding as in the proof of [L-T.7, Lemma 3.3] from (A.5) to (A.15) in Appendix A of this reference.

Step 4. Using identity (6.48), and the estimates (6.51) and (6.60), we obtain (6.47). ∎

The proof of Lemma 6.11 is complete. ∎

Lemma 6.11 allows us to complete the estimate of the first integral term on the right of (6.35), hence of the desired integral term of Theorem 6.6.

COROLLARY 6.12. *Under the assumptions of Lemma* 6.11, *we have*

$$\int_0^T (A^{1/2} w, h \cdot \nabla(D\, D^* A^{1/2} w_t))_\Omega \, dt = \mathcal{O}\left(\int_0^T \|D^* A^{1/2} w_t\|_{L_2(\Gamma)} \|A^{1/2} w\|_{L_2(\Omega)} \, dt\right) + \mathcal{O}(E_w(T) + E_w(0)), \tag{6.62}$$

where the constants in $\mathcal{O}$ depend on $\|D\|$, b, but not on T.

Proof. From (6.47) by integration by parts in t,

$$\begin{aligned}\int_0^T & (A^{1/2} w(t), h \cdot \nabla(D\, D^* A^{1/2} w_t(t)))_\Omega \, dt \\ &= +\frac{1}{2}\left[\left(\frac{\partial}{\partial \nu} D(b\, D^* A^{1/2} w(t)), D^* A^{1/2} w(t)\right)_\Gamma\right]_0^T \\ &\quad + \mathcal{O}\left(\int_0^T \|D^* A^{1/2} w_t\|_{L_2(\Gamma)} \|A^{1/2} w\|_{L_2(\Omega)} \, dt\right). \end{aligned} \tag{6.63}$$

Now $A^{1/2} w(t) \in L_2(\Omega)$ implies $D^* A^{1/2} w(t) \in H^{1/2}(\Gamma)$ by (2.14) with $s = 0$ and with b smooth, we have that $D(b\, D^* A^{1/2} w(t)) \in H^1(\Omega)$ by (2.13), and it solves the Laplace equations. Therefore $(\partial/\partial \nu)\, D(b\, D^* A^{1/2} w(t)) \in H^{-1/2}(\Gamma)$ [K.1, Theorem 3.8.1, p. 71 and ff]. Thus

$$\left| \left(\frac{\partial}{\partial \nu} D(b\, D^* A^{1/2} w(t)),\, D^* A^{1/2} w(t) \right)_\Gamma \right|$$
$$\leqslant \left\| \frac{\partial}{\partial \nu} D(b\, D^* A^{1/2} w(t)) \right\|_{H^{-1/2}(\Gamma)} \|D^* A^{1/2} w(t)\|_{H^{1/2}(\Gamma)}$$
$$\leqslant C\, \|A^{1/2} w(t)\|^2_{L_2(\Omega)} \leqslant C E_w(t). \tag{6.64}$$

Thus, (6.64) used in (6.63) yields (6.62). ∎

To complete the proof of Theorem 6.6, we combine (6.62) with (6.35), thus obtaining (6.27). ∎

Appendix A

For future reference to regularity/exact controllability/uniform stabilization problems for Eq. (1.1a) with boundary conditions different from (1.1c)–(1.1d), we shall first derive a general identity for ϕ only solution to (1.2a), with no use of boundary conditions (1.2c)–(1.2d), in terms of a general smooth vector field $h(x) = [h_1(x), ..., h_n(x)]$ over $\bar{\Omega}$, see Eq. (A.5) below. Next, we shall specialize such an identity (A.5) to ϕ which satisfies also the B.C. (1.2c)–(1.2d) of this paper.

Identity for ϕ Which Solves (1.2a). We multiply (1.2a) by $h \cdot \nabla(\Delta\phi)$ and integrate over Q. We obtain, see respectively [L-T.4, Eq. (A.6) and Eq. (A.7), Appendix A]

$$\int_Q \phi_{tt} h \cdot \nabla(\Delta\phi)\, dQ$$
$$= [(\phi_t, h \cdot \nabla(\Delta\phi))_\Omega]_0^T - \int_\Sigma \phi_t\, \Delta\phi_t h \cdot \nu\, d\Sigma - \frac{1}{2} \int_\Sigma |\nabla\phi_t|^2\, h \cdot \nu\, d\Sigma$$
$$+ \int_\Sigma \frac{\partial \phi_t}{\partial \nu} h \cdot \nabla\phi_t\, d\Sigma + \int_\Sigma \frac{\partial \phi_t}{\partial \nu} \phi_t \operatorname{div} h\, d\Sigma - \int_Q H\, \nabla\phi_t \cdot \nabla\phi_t\, dQ$$
$$- \frac{1}{2} \int_Q |\nabla\phi_t|^2 \operatorname{div} h\, dQ - \int_Q \phi_t\, \nabla(\operatorname{div} h) \cdot \nabla\phi_t\, dQ \tag{A.1}$$

$$\int_Q \Delta^2 \phi h \cdot \nabla(\Delta\phi)\, dQ$$
$$= \int_\Sigma \frac{\partial(\Delta\phi)}{\partial \nu} h \cdot \nabla(\Delta\phi)\, d\Sigma - \frac{1}{2} \int_\Sigma |\nabla(\Delta\phi)|^2\, h \cdot \nu\, d\Sigma$$
$$- \int_Q H\, \nabla(\Delta\phi) \cdot \nabla(\Delta\phi)\, dQ + \frac{1}{2} \int_Q |\nabla(\Delta\phi)|^2 \operatorname{div} h\, dQ. \tag{A.2}$$

Finally, we compute the new term by integration by parts on t

$$\gamma \int_Q \Delta\phi_{tt} h \cdot \nabla(\Delta\phi)\, dQ$$
$$= \gamma[(\Delta\phi_t, h \cdot \nabla(\Delta\phi))_\Omega]_0^T$$
$$-\frac{\gamma}{2}\int_\Sigma (\Delta\phi_t)^2\, h \cdot \nu\, d\Sigma + \frac{\gamma}{2}\int_Q (\Delta\phi_t)^2 \operatorname{div} h\, dQ \tag{A.3}$$

after using also the identity

$$\int_\Omega h \cdot \nabla\psi\, d\Omega = \int_\Gamma \psi h \cdot \nu\, d\Gamma - \int_\Omega \psi \operatorname{div} h\, d\Omega \tag{A.4}$$

with $\psi = \Delta\phi_t$ and h replaced by $\Delta\phi_t h$. Using (A.1)–(A.3) in Eq. (1.2a) results in the identity

$$\int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu} h \cdot \nabla(\Delta\phi)\, d\Sigma + \int_\Sigma \frac{\partial\phi_t}{\partial\nu} h \cdot \nabla\phi_t\, d\Sigma$$
$$+\frac{\gamma}{2}\int_\Sigma (\Delta\phi_t)^2\, h \cdot \nu\, d\Sigma - \frac{1}{2}\int_\Sigma |\nabla(\Delta\phi)|^2\, h \cdot \nu\, d\Sigma$$
$$-\frac{1}{2}\int_\Sigma |\nabla\phi_t|^2\, h \cdot \nu\, d\Sigma + \int_\Sigma \frac{\partial\phi_t}{\partial\nu} \phi_t \operatorname{div} h\, d\Sigma - \int_\Sigma \phi_t\, \Delta\phi_t h \cdot \nu\, d\Sigma$$
$$= \int_Q H\,\nabla(\Delta\phi) \cdot \nabla(\Delta\phi)\, dQ + \int_Q H\,\nabla\phi_t \cdot \nabla\phi_t\, dQ$$
$$+\frac{1}{2}\int_Q \{|\nabla\phi_t|^2 + \gamma(\Delta\phi_t)^2 - |\nabla(\Delta\phi)|^2\} \operatorname{div} h\, dQ$$
$$+\int_Q \phi_t \nabla(\operatorname{div} h) \cdot \nabla\phi_t\, dQ + \int_Q fh \cdot \nabla(\Delta\phi)\, dQ$$
$$-[(\phi_t, h \cdot \nabla(\Delta\phi))_\Omega + \gamma(\Delta\phi_t, h \cdot \nabla(\Delta\phi))_\Omega]_0^T. \tag{A.5}$$

Specialization of the Left Hand Side of (A.5) *to* ϕ *Satisfying Also the Boundary Conditions* (1.2c)–(1.2d). Recalling (1.2c)–(1.2d), we obtain on

$$\Sigma\colon \phi_t \equiv 0; \qquad \Delta\phi_t \equiv 0; \qquad \nabla\phi_t \perp \Gamma; \qquad \nabla(\Delta\phi) \perp \Gamma$$
$$h \cdot \nabla\phi_t = \frac{\partial\phi_t}{\partial\nu} h \cdot \nu; \qquad |\nabla\phi_t| = \left|\frac{\partial\phi_t}{\partial\nu}\right| \tag{A.6}$$
$$h \cdot \nabla(\Delta\phi) = \frac{\partial(\Delta\phi)}{\partial\nu} h \cdot \nu; \qquad |\nabla(\Delta\phi)| = \left|\frac{\partial(\Delta\phi)}{\partial\nu}\right|.$$

Thus, the left hand side (L.H.S.) of identity (A.5) becomes

$$\text{L.H.S. of (A.5)} = \frac{1}{2}\int_{\Sigma}\left[\left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 + \left(\frac{\partial\phi_t}{\partial\nu}\right)^2\right\} h\cdot\nu\, d\Sigma. \tag{A.7}$$

Specialization of the Right Hand Side of (A.5) *to Radial Vector Fields* $h(x) = x - x_0$. Using $\operatorname{div} h \equiv \dim \Omega = n$; $H(x) \equiv$ identity, we obtain for the right hand side of (A.5)

$$\begin{aligned}\text{R.H.S. of (A.5)} = {} & \int_Q |\nabla(\Delta\phi)|^2 + |\nabla\phi_t|^2\, dQ + \frac{n}{2}\int_Q |\nabla\phi_t|^2 - |\nabla(\Delta\phi)|^2\, dQ \\ & + \frac{n\gamma}{2}\int_Q (\Delta\phi_t)^2\, dQ + \int_Q fh\cdot\nabla(\Delta\phi)\, dQ \\ & - [(\phi_t, h\cdot\nabla(\Delta\phi))_\Omega + \gamma(\Delta\phi_t, h\cdot\nabla(\Delta\phi))_\Omega]_0^T. \end{aligned} \tag{A.8}$$

Appendix B

Again, we shall first obtain an identity, (B.4) below, for ϕ that solves only (1.2a) and for an arbitrary smooth vector field $h(x)$ on $\bar{\Omega}$. Next, we shall specialize this identity (B.4) to the case where ϕ satisfies, in addition, the B.C. (1.2c)–(1.2d) and, moreover, the vector field is radial.

Identity for ϕ Which Solves (1.2a). We multiply Eq. (1.2a) by $\Delta\phi \operatorname{div} h$ and integrate over Q. We obtain, see respectively [L-T.4, Eq. (B.1), and Eq. (B.2), Appendix B]

$$\begin{aligned}\int_Q \phi_{tt}\,\Delta\phi \operatorname{div} h\, dQ = {} & \left[\int_\Gamma \frac{\partial\phi}{\partial\nu}\phi_t \operatorname{div} h\, d\Gamma = \int_\Omega \nabla\phi\cdot\nabla(\phi_t \operatorname{div} h)\, d\Omega\right]_0^T \\ & - \int_\Sigma \frac{\partial\phi_t}{\partial\nu}\phi_t \operatorname{div} h\, d\Sigma + \int_Q |\nabla\phi_t|^2 \operatorname{div} h\, dQ \\ & + \int_Q \phi_t\nabla(\operatorname{div} h)\cdot\nabla\phi_t\, dQ \end{aligned} \tag{B.1}$$

$$\begin{aligned}\int_Q \Delta^2\phi\,\Delta\phi \operatorname{div} h\, dQ = {} & \int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu}\Delta\phi \operatorname{div} h\, d\Sigma - \int_Q |\nabla(\Delta\phi)|^2 \operatorname{div} h\, dQ \\ & - \int_Q \Delta\phi\,\nabla(\operatorname{div} h)\cdot\nabla(\Delta\phi)\, dQ. \end{aligned} \tag{B.2}$$

Finally, the new term is

$$\gamma\int_Q \Delta\phi_{tt}\,\Delta\phi \operatorname{div} h\, dQ = [\gamma(\Delta\phi_t, \Delta\phi \operatorname{div} h)_\Omega]_0^T - \gamma\int_Q (\Delta\phi_t)^2 \operatorname{div} h\, dQ. \tag{B.3}$$

Using (B.1)–(B.3) in Eq. (1.2a) yields

$$\int_Q \{|\nabla\phi_t|^2+\gamma(\Delta\phi_t)^2-|\nabla(\Delta\phi)|^2\}\operatorname{div} h\, dQ$$

$$=\int_\Sigma \frac{\partial\phi_t}{\partial\nu}\phi_t \operatorname{div} h\, d\Sigma-\int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu}\Delta\phi\operatorname{div} h\, d\Sigma$$

$$+\int_Q f\,\Delta\phi\operatorname{div} h\, dQ+\int_Q \Delta\phi\,\nabla(\operatorname{div} h)\cdot\nabla(\Delta\phi)\, dQ$$

$$-\int_Q \phi_t\nabla(\operatorname{div} h)\cdot\nabla\phi_t\, dQ$$

$$+\left[\int_\Omega \nabla\phi\cdot\nabla(\phi_t\operatorname{div} h)\, d\Omega-\int_\Gamma \frac{\partial\phi}{\partial\nu}\phi_t\operatorname{div} h\, d\Gamma\right]_0^T$$

$$+\gamma[(\Delta\phi_t,\Delta\phi\operatorname{div} h)_\Omega]_0^T. \tag{B.4}$$

Specialization of (B.4) *to* ϕ *Satisfying Also the B.C.* (1.2c)–(1.2d). From (1.2c)–(1.2d) we obtain for future reference

$$\frac{1}{2}\int_Q \{|\nabla\phi_t|^2+\gamma(\Delta\phi_t)^2-|\nabla(\Delta\phi)|^2\}\operatorname{div} h\, dQ$$

$$=\frac{1}{2}\int_Q \Delta\phi\,\nabla(\operatorname{div} h)\cdot\nabla(\Delta\phi)\, dQ-\frac{1}{2}\int_Q \phi_t\nabla(\operatorname{div} h)\cdot\nabla\phi_t\, dQ$$

$$+\frac{1}{2}\int_Q f\,\Delta\phi\operatorname{div} h\, dQ+\frac{1}{2}[\gamma(\Delta\phi_t,\Delta\phi\operatorname{div} h)_\Omega-(\phi_t,\Delta\phi\operatorname{div} h)_\Omega]_0^T. \tag{B.5}$$

In particular, if we use more simply the multiplier $\Delta\phi$ in the above procedure, we obtain

$$\int_Q \{|\nabla\phi_t|^2+\gamma(\Delta\phi_t)^2-|\nabla(\Delta\phi)|^2\}\, dQ$$

$$=[(\gamma\,\Delta\phi_t-\phi_t,\Delta\phi)_\Omega]_0^T+\int_Q f\,\Delta\phi\, dQ \tag{B.6}$$

to be used in (4.8) and (6.23).

Appendix C

Recalling A and $A^{1/2}$ from (1.3), we see that the abstract version of the homogeneous problem (1.2) is

$$\phi_{tt}+\gamma A^{1/2}\phi_{tt}+A\phi=f,$$

or

$$\phi_{tt} = -(I+\gamma A^{1/2})^{-1} A\phi + (I+\gamma A^{1/2})^{-1} f \tag{C.1}$$

$\gamma > 0$. But

$$A^{1/2}(I+\gamma A^{1/2})^{-1} = \frac{1}{\gamma} I - \frac{1}{\gamma}(I+\gamma A^{1/2})^{-1} \qquad \text{on } L_2(\Omega), \tag{C.2}$$

and thus, recalling $\mathscr{A}$ from (2.2), we have on $\mathscr{D}(A^{1/2})$

$$-\mathscr{A} = (I+\gamma A^{1/2})^{-1} A = \frac{A^{1/2}}{\gamma} - \frac{1}{\gamma}(I+\gamma A^{1/2})^{-1} A^{1/2} \tag{C.3}$$

$$= \frac{A^{1/2}}{\gamma} - \frac{1}{\gamma^2} I + \frac{1}{\gamma^2}(I+\gamma A^{1/2})^{-1}, \tag{C.4}$$

where in going from (C.3) to (C.4) we have used (C.2). Thus, on $\mathscr{D}(A^{1/2})$, the generator $\mathscr{A}$ of the second-order equation (C.1) acts like a bounded perturbation of $A^{1/2}/\gamma = -\Delta/\gamma$, which reveals the hyperbolic character of Eq. (1.2) with speed of propagation $\sqrt{1/\gamma}$. Thus, (1.2) can be rewritten as

$$\begin{cases} \phi_{tt} = \dfrac{1}{\gamma}\Delta\phi + \dfrac{\phi}{\gamma^2} - (I+\gamma A^{1/2})^{-1}\phi + (I+\gamma A^{1/2})^{-1} f; \\ \phi|_{\Sigma} = \Delta\phi|_{\Sigma} = 0. \end{cases} \tag{C.5}$$

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